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Synchronization of chaotic lasers using an active control

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In this work we present a possibility to realise the synchronization between two identical chaotic lasers by using an active control. We describe the action of the synchronization mechanism generally and then we apply it in the case of Lorenz-Robbins-Haken system. The numerical simulations of the analytical results are presented.

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1. Introduction

It is now well known that the laser, considered a device that exemplifies the notions of stability, coherence and purity, can behave chaotic. While many works have been devoted to eliminate such behaviour, there is an increasing interest to exploit, rather than prevent, chaos in lasers. The essential idea is of using chaotic laser signals in communications. The signals produced by a chaotic system, such as by a chaotic laser, are highly complex and contain a broad range of frequencies. The information transmitted is thus hidden within these noisy signals. Hence a chaotic laser system can be used to synchronize a transmitter and receiver, and to transmit encrypted data [1].

The essential characteristic of chaotic systems is their great sensitive dependence on the initial conditions. Consequently any small initial disturbance will be amplified and the corresponding trajectories of the system will diverge exponentially from each other. The possibility to synchronize two identical chaotic systems, which starts from slightly different initial conditions, seems unlikely. However, it has been suggested [2] and latter proved [3]-[12] that chaotic systems may be synchronized in some circumstances. During the last two decades synchronization in chaotic dynamic systems has received a great interest among scientists from various research fields due to its practical applications in laser dynamics [13], chemical and biological systems [14], electronic circuits [9], [15] or secure communication [16].

The idea of synchronization is to use the output of the master system to control the slave system so that the output of the response system follows the output of the master system asymptotically.

Since the seminal contribution by Pecora and Carroll [3], a large number of investigations have been made and different techniques have been applied in simulations and experiments. Thus rigorous results have been obtained using Lyapunov functions [4],[17],[18], but the method can be applied only to particular examples. Another rigorous technique, which implies to show that all normal Lyapunov exponents are negative for all measures of the dynamics, is that of Ashwin et all [19]. Unfortunately this leads to an intensive numerical analysis for typical dynamical systems. There are also the approaches based on the PGY idea of controlling chaos [20], or on the open-plus-close-loop (OPCL) control method [21].

The approach we present in this work examines the synchronization between two identical chaotic laser systems. The systems are described by the autonomous flows, i.e. the Lorenz-Haken equations [22], and the synchronization is realised with a technique used in active control theory [23], [24]. First we describe the action of the synchronization mechanism generally, and then the particular application to the laser equations is illustrated.

2. Synchronization method

Let us consider two identical chaotic flows, described by the equation:

$$\frac{d\vec{x}(t,\lambda_i)}{dt} = \vec{F}[\vec{x}(t,\lambda_i)] \qquad (1)$$

where $\vec{x} \in \Re^n$, λ_i (i=1,2..) are parameters and $\vec{F}$ is a nonlinear function. We will consider only the chaotic phase trajectories of the system (1) and furthermore the subscripts 1 and 2, will differentiate the master and the slave systems, respectively. Thus the systems can be written:

$$\begin{aligned} \frac{d\vec{x}_1(t,\lambda_i)}{dt} &= \vec{F}_1[\vec{x}_1(t,\lambda_i)] \\ \frac{d\vec{x}_2(t,\lambda_i)}{dt} &= \vec{F}[\vec{x}_2(t,\lambda_i)] + \vec{G}(t) \end{aligned} \quad , \qquad (2)$$

where $\vec{x}_1, \vec{x}_2 \in \Re^n$ and $\vec{G} \in \Re^n$ are active control functions that are to be determined.

Let us define the state error between the response system that is to be controlled and the controlling drive system as: $\vec{e}(t,\lambda_i) = \vec{x}_2(t,\lambda_i) - \vec{x}_1(t,\lambda_i)$. After substraction of the relations (2), we have:

$$\frac{d\vec{e}(t)}{dt} = \vec{F}[\vec{e}(t)] + \vec{R}[\vec{x}_1(t), \vec{x}_2(t)] + \vec{G}(t) \qquad (3)$$

where for simplicity the notation of parameters is omitted.

Now we define the active control function $\vec{G}(t)$ as

$$\vec{G}(t) = \vec{L}(t) - \vec{R}(\vec{x}_1, \vec{x}_2) \qquad (4)$$

so that

$$\frac{d\vec{e}(t)}{dt} = \vec{F}[\vec{e}(t)] + \vec{L}(t)\,. \qquad (5)$$

This equation describes the error dynamics and can be considered in terms of a control problem, where the system to be controlled is a linear one. With a control input

$$\vec{L}(t) = M.\vec{e}(t)$$

where M is a matrix (n x n), we obtain

$$\frac{d\vec{e}(t)}{dt} = A.\vec{e}(t) \qquad (6)$$

If the eigenvalues of the matrix A are smaller than zero, the difference between the two systems converge to zero as t goes to infinity. Consequently the synchronization of the considered systems is realized.

3. Application of the method to the chaotic lasers

As a particular application of the method we have chosen the synchronization of two identical Lorenz-Robbins-Haken systems, which are nonlinear dynamical systems with a rich dynamical behaviour, even a chaotic one, when the parameters are varied [25].

The laser equations are:

$$\begin{aligned}\frac{dE}{dt} &= k(P - E)\\ \frac{dP}{dt} &= \gamma_{\perp}(ED - P)\\ \frac{dD}{dt} &= \gamma_{\rm II}[(\lambda + 1) - D - \lambda EP]\end{aligned} \quad , \qquad (7)$$

where E, P and D are the normalized electric field strength, the polarization and the atomic inversion, respectively. The significance of the parameters in the system (1) is: k- the cavity loss; $\gamma_{\perp}$ – the linewidth; $\gamma_{\rm II}$ - the inverse longitudinal relaxation time, and $\lambda = \frac{D_0 - D_c}{D_c}$, where D_0 and D_c are the unsaturated and critical inversion, respectively, represents the pump power.

The classical Lorenz equations are:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x)\\ \frac{dy}{dt} &= rx - y - xz\\ \frac{dz}{dt} &= xy - bz\end{aligned} \quad , \qquad (8)$$

where the variable x is proportional to the circulatory fluid flow velocity, y characterizes the temperature difference between rising and falling fluid regions, and z characterizes the distortion of the vertical temperature profile from its linear, with height, equilibrium variation. The dimensionless parameters in the equations (2) have the following significance: σ > 0 is the Prandtl number, $r = \frac{R}{R_c}$, with R- the Rayleigh number, is considered as the control parameter, and b > 0.

If r » σ, Robbins [26] obtained a modified Lorenz model without losing its chaotic behaviour. Let consider the system of equations (8) with the following substitution of variables:

$$\begin{aligned} z &= r - u\\ x &\to y\\ y &\to z\end{aligned} \quad , \qquad (9)$$

then, with a convenient notation, $y \to X$, $z \to Y$, $u \to Z$,we obtain:

$$\begin{aligned}\frac{dX}{dt} &= \sigma(Y - X)\\ \frac{dY}{dt} &= XZ - Y\\ \frac{dZ}{dt} &= b(r - Z) - XY\end{aligned} \quad . \qquad (10)$$

It can be seen that the system (10) is identical with the system (7), with the following identification:

$$t \to t\frac{\sigma}{k};\ E \to \frac{X}{\sqrt{b(r-1)}};\ P \to \frac{Y}{\sqrt{b(r-1)}};$$
$$D \to Z;\ \gamma_{\rm II} = \frac{kb}{\sigma};\ \gamma_{\perp} = \frac{k}{\sigma};\ \lambda = r - 1 \ .$$

For simplicity, we will write the system (10) in the form:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= xz - y\\ \dot{z} &= b(r - z) - xy\end{aligned} \quad , \qquad (11)$$

where the dot represents the derivation with respect to the new time $t\frac{\sigma}{k}$, and we will name (11) the Lorenz-Robbins-Haken (LRH) system.

Thus the systems under synchronization are:

$$\begin{aligned}\dot{x}_1 &= \sigma(y_1 - x_1)\\ \dot{y}_1 &= x_1 z_1 - y_1\\ \dot{z}_1 &= b(r - z_1) - x_1 y_1\end{aligned} \tag{12}$$

and

$$\begin{aligned}\dot{x}_2 &= \sigma(y_2 - x_2) + g_x(t)\\ \dot{y}_2 &= x_2 z_2 - y_2 + g_y(t) \quad .\\ \dot{z}_2 &= b(r - z_2) - x_2 y_2 + g_z(t)\end{aligned} \tag{13}$$

Subtracting the first system (12) from the second system (13) which includes the control signals, we obtain

$$\begin{aligned}\dot{e}_x &= \sigma(e_y - e_x) + g_x(t)\\ \dot{e}_y &= x_2 z_2 - x_1 z_1 - e_y + g_y(t) \quad ,\\ \dot{e}_z &= -b e_z - x_2 y_2 + x_1 y_1 + g_z(t)\end{aligned} \tag{14}$$

where $e_x = x_2 - x_1$; $e_y = y_2 - y_1$ and $e_z = z_2 - z_1$.

Now we define the active control functions as

$$\begin{cases} g_x(t) = g_1(t)\\ g_y(t) = g_2(t) - x_2 z_2 + x_1 z_1\\ g_z(t) = g_3(t) + x_2 y_2 - x_1 y_1\end{cases} \tag{15}$$

The substitution of Eq.(15) in (14) leads to:

$$\begin{bmatrix}\dot{e}_x\\ \dot{e}_y\\ \dot{e}_z\end{bmatrix} = \begin{bmatrix}-\sigma & \sigma & 0\\ 0 & -1 & 0\\ 0 & 0 & -b\end{bmatrix}\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}g_1\\ g_2\\ g_3\end{bmatrix}. \tag{16}$$

Equation (16) describes the error dynamics and can be considered in terms of a control problem where the system to be controlled is now a linear system with the control input $g(t) = [g_1, g_2, g_3]^T$. Since the error dynamics is full state controllable, the feedback gains can be designed to stabilize the state of the error system $[e_x, e_y, e_z]^T$ so that the error signal converge to zero as time t goes to infinity. This implies that the two chaotic systems are synchronized. There are many possible choices for the controller g (t). We choose

$$\begin{bmatrix}g_1\\ g_2\\ g_3\end{bmatrix} = -\begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix}\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix} \tag{17}$$

where the constants α_{ij}, which are the controller gains, must be considered so that the characteristic matrix of the closed loop system to have all the eigenvalues with negative real parts. If we consider

$$\begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix} = \begin{bmatrix}-\sigma+1 & \sigma & 0\\ 0 & 0 & 0\\ 0 & 0 & -b+1\end{bmatrix} \tag{18}$$

the eigenvalues will be -1, -1, -1, and as we will observe in numerical investigations, this lead to the synchronization of the two LRH systems.

4. Numerical verifications

In order to demonstrate the veracity of this synchronization method for chaotic lasers, we have chose the values of the parameters in the chaotic domain, i.e. $\sigma = 2$, b=1/4, and $r = 30$. As can be observed from the Figures1and 2, the trajectories of two LRH systems, which start from different initial conditions are finally synchronized. In these figures the signals $x_1(t)$, $y_1(t)$, $z_1(t)$ and $x_2(t)$, $y_2(t)$, $z_2(t)$ with the control deactivated and activated are illustrated. In Fig. 3 it can be seen the dynamics of the error between the two lasers signals $x_1(t)$ and $x_2(t)$, when the control is activated.

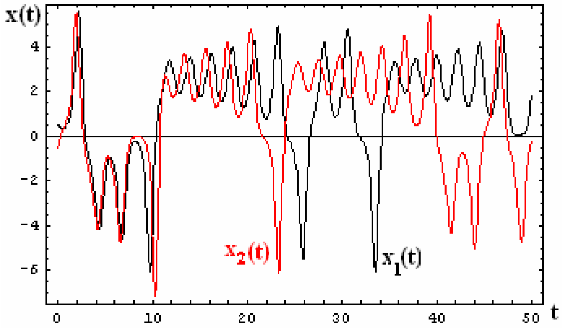

Fig. 1(a). The signals $x_1(t)$ and $x_2(t)$ with the initial conditions 0.5 and -0.5, respectively.

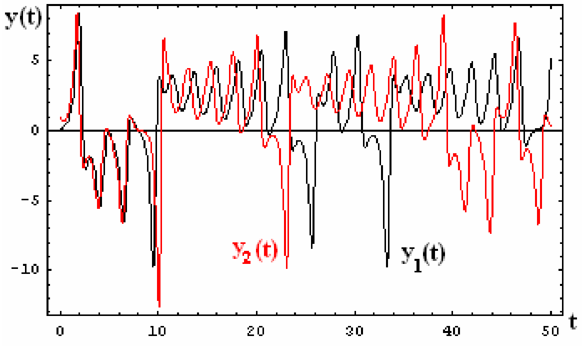

Fig. 1(b). The signals y_1 (t) and $y_2(t)$ with the initial conditions 0.1 and 1, respectively.

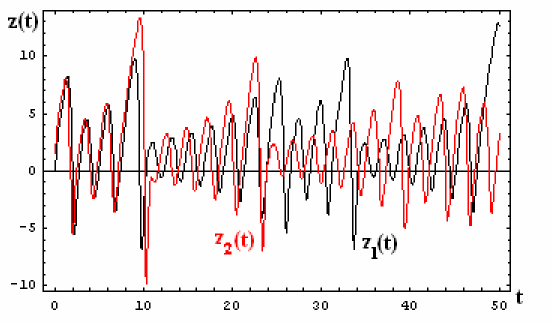

Fig. 1(c). The signals z_1 (t) and z_2(t) with the initial conditions 0.1 and 1.5, respectively.

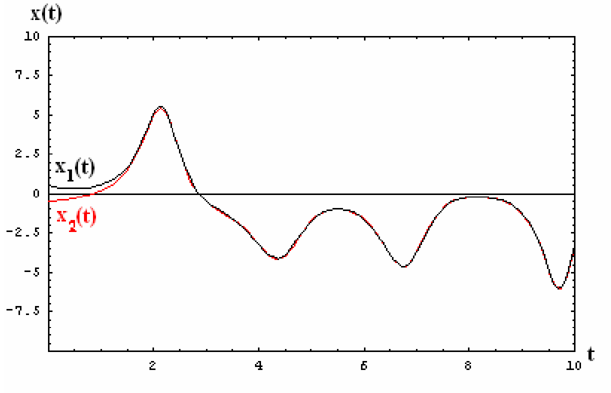

Fig. 2 (a). The signals x_1(t) and x_2(t) after synchronization. The initial conditions are the same as in the Fig. 1(a).

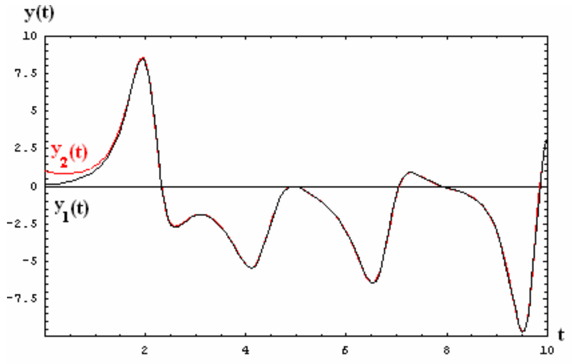

Fig. 2 (b). The signals y_1(t) and y_2(t) after synchronization. The initial conditions are the same as in the Fig. 1(b).

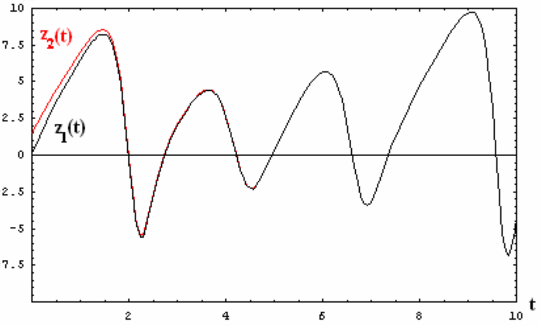

Fig. 2 (c). The signals z_1(t) and z_2(t) after synchronization. The initial conditions are the same as in the Fig. 1(c).

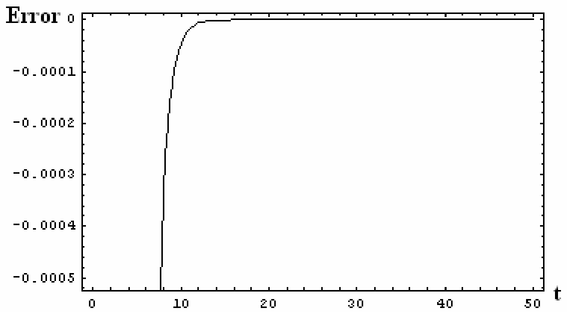

Fig. 3. Dynamics of the error e_x, between x_1(t) and x_2(t), with the control activated.

5. Conclusions

This work demonstrates that chaotic trajectories in two laser systems can be synchronized using a method of active control. This method has been used by us also in the case of two chaotic maps [27]. We realized even an indirect synchronization of two chaotic maps using a third one [28], and also a synchronization of two spatiotemporal nonlinear dynamical systems [29]. The synchronization of chaotic lasers has been observed not only in numerical models but also in laser experiments [30]-[32].

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