Stationary optical solitons with Kudryashov's self–phase modulation and nonlinear chromatic dispersion

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This paper recovers stationary optical soliton solutions to the nonlinear Schrödinger's equation that maintains Kudryashov's law of refractive index. Both linear temporal evolution and generalized temporal evolution are considered. In both cases, the results are in terms of Appell's hypergeometric function.

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1. Introduction

The theory of optical solitons have sculpted the telecommunications industry that is an engineering marvel today. The two key features in its governing dynamics are chromatic dispersion (CD) and the nonlinear law of refractive index that is also known as self–phase modulation (SPM). The delicate balance between CD and SPM is what makes the solitons sustain for fiber optic communication. This is the technological know–how of Internet communication dynamics [1–10]. There are various forms of known SPM that are applicable to a variety of fiber materials. Very recently, Nikolay Kudryashov from Moscow, Russia has proposed the well–known Lakshmanan–Porsezian–Daniel equation with arbitrary for of refractive index [5]. There are other forms of self–phase modulation (SPM) that was proposed by him earlier. Today's paper will address one such form of SPM, as proposed by Kudryashov, which comes with four nonlinear terms having power law format. Incidentally, CD also has a nonlinear structure that is attributed to various manifestations of the optical fiber including its rough handling during underground and/or undersea installation. Thus, the governing model which is the nonlinear Schrödinger's equation (NLSE) will be studied with nonlinear CD and Kudryashov's quadrupled form of power law of nonlinear SPM. The stationary soliton solutions to the NLSE, with Kudryashov's law of nonlinearity will be addressed in this work for linear temporal evolution as well as generalized temporal evolution of the pulses. It must be noted that stationary solitons to additional forms of NLSE with non–Kerr laws of nonlinearity as well as other governing models, from nonlinear optics, have been reported. They are Lakshmanan–Porsezian–Daniel model as well as Sasa–Satsuma equation [1, 2]. The detailed mathematical formulation and the solution strategy, of today's model, are derived and exhibited after a quick and succinct intro.

1.1. Governing model

The governing NLSE with Kudryashov's form of nonlinear refractive index, and linear CD, was proposed as [4]

$$iq_t + aq_{xx} + \left(\frac{b_1}{|q|^{2n}} + \frac{b_2}{|q|^n} + b_3|q|^n + b_4|q|^{2n} \right) q = 0. \qquad (1)$$

In equation (1), the linear temporal evolution is given by the first term where $i = \sqrt{-1}$ while a is the coefficient of chromatic dispersion (CD). The four nonlinear terms that come from the coefficients of b_j for $j = 1,2,3,4$ account for nonlinear form of refractive index as proposed by Kudryashov during 2019 [4]. Several results including soliton solutions and its conservation laws have been recovered for such a model. The subsequent section first remodels (1) for nonlinear CD and its further analysis is carried out.

2. Stationary solitons

The NLSE with Kudryashov's form of refractive index for nonlinear CD will be addressed first for linear temporal evolution followed by the generalized temporal evolution. This study would therefore be split into the following two subsequent subsections.

2.1. Linear temporal evolution

The governing NLSE with nonlinear CD having Kudryashov's form of nonlinear refractive index is structured as [1, 2]

$$iq_t + a(|q|^m q)_{xx} + \begin{pmatrix} \frac{b_1}{|q|^{2n}} + \frac{b_2}{|q|^n} \\ +b_3|q|^n \\ +b_4|q|^{2n} \end{pmatrix} q = 0. \tag{2}$$

The parameter m accounts for nonlinear CD. For $m = 0$, CD is rendered to be linear, in which case equation (2) collapses to (1). To solve (2), the decomposition of $q(x,t)$ into phase–amplitude format is carried out as

$$q(x,t) = \phi(x)e^{i\lambda t}. \tag{3}$$

Substituting (3) into (2) yields

$$\begin{gathered} am(m+1)\{\phi'(x)\}^2\phi^{m+2n}(x) \\ +a(m+1)\phi''(x)\phi^{m+2n+1}(x) + b_1\phi^2(x) \\ +b_2\phi^{n+2}(x) + b_3\phi^{3n+2}(x) + b_4\phi^{4n+2}(x) \\ -\lambda\phi^{2n+2}(x) = 0. \end{gathered} \tag{4}$$

For integrability purposes, the choices

$$b_2 = 0,\ b_3 = 0, \tag{5}$$

are made. Equation (2) therefore simplifies to

$$iq_t + a(|q|^m q)_{xx} + \begin{pmatrix} \frac{b_1}{|q|^{2n}} \\ +b_4|q|^{2n} \end{pmatrix} q = 0, \tag{6}$$

while (4) reduces to

$$\begin{gathered} am(m+1)\{\phi'(x)\}^2\phi^{m+2n}(x) \\ +a(m+1)\phi''(x)\phi^{m+2n+1}(x) + b_4\phi^{4n+2}(x) \\ +b_1\phi^2(x) - \lambda\phi^{2n+2}(x) = 0. \end{gathered} \tag{7}$$

The above equation admits a single Lie point symmetry, namely $\partial/\partial x$. This symmetry will be applied in the integration process. Integrating and discarding the constants of integration, the following implicit solution is yielded:

$$x = \sqrt{B_3}F_1\begin{pmatrix} \frac{1}{4}\left(\frac{m}{n}+2\right); \\ \frac{1}{2},\frac{1}{2}; \\ \frac{1}{4}\left(\frac{m}{n}+6\right); \\ B_1, B_2 \end{pmatrix}, \tag{8}$$

where

$$B_1 = \frac{2b_4(m+2)(m-2n+2)\phi^{2n}}{\lambda(m-2n+2)(m+2n+2)-P}, \tag{9}$$

$$P = \sqrt{\begin{matrix} \{(m+2)^2 - 4n^2\} \\ \times\begin{Bmatrix} \lambda^2(m-2n+2)(m+2n+2) \\ -4b_1b_4(m+2)^2 \end{Bmatrix} \end{matrix}},$$

$$B_2 = \frac{2b_4(m+2)(m-2n+2)\phi^{2n}}{Q+\lambda(m-2n+2)(m+2n+2)}, \tag{10}$$

$$Q = \sqrt{\begin{matrix} \{(m+2)^2 - 4n^2\} \\ \times\begin{Bmatrix} \lambda^2(m-2n+2)(m+2n+2) \\ -4b_1b_4(m+2)^2 \end{Bmatrix} \end{matrix}},$$

and

$$B_3 = -\frac{2a(m+1)(m-2n+2)\phi^{m+2n}}{b_1(m+2n)^2}. \tag{11}$$

Here, Appell's hypergeometric function is written as

$$\begin{gathered} F_1(\alpha;\beta_1,\beta_2;\gamma;u,v) \\ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{m!n!(\gamma)_{m+n}}u^m v^n, \end{gathered} \tag{12}$$

where the Pochhammer symbol is:

$$(p)_n = \begin{cases} 1, & n = 0, \\ p(p+1)\cdots(p+n-1), & n > 0. \end{cases} \tag{13}$$

The hypergeometric function in (12) is defined for

$$|u| < 1,\ |v| < 1. \tag{14}$$

For (8), this would imply

$$\phi(x) < \left|\frac{\lambda(m-2n+2)(m+2n+2)-R}{2b_4(m+2)(m-2n+2)}\right|^{\frac{1}{2n}}, \tag{15}$$

with

$$R = \sqrt{\begin{matrix} \{(m+2)^2 - 4n^2\} \\ \times\begin{Bmatrix} \lambda^2(m-2n+2)(m+2n+2) \\ -4b_1b_4(m+2)^2 \end{Bmatrix} \end{matrix}}.$$

Moreover, from (8) and (11), one would conclude the necessary requirement

$$ab_1 < 0. \tag{16}$$

This is another constraint on the parameters of the governing equation that is needed for stationary solitons.

2.2. Generalized temporal evolution

The model for NLSE with nonlinear CD and Kudryashov's form of SPM having generalized temporal evolution is structured as

$$i(q^l)_t + a(|q|^m q^l)_{xx} + \begin{pmatrix} \frac{b_1}{|q|^{2n}} + \frac{b_2}{|q|^n} \\ +b_3|q|^n \\ +b_4|q|^{2n} \end{pmatrix} q^l = 0. \tag{17}$$

In (17), the parameter l accounts for generalized temporal evolution. Thus, for $l = 1$, (17) collapses to (2). Next, decomposing $q(x,t)$ into phase–amplitude format as in (3), and substituting into (17) gives:

$$\begin{aligned} &a(l+m)(l+m-1)\{\phi'(x)\}^2\phi^{m+2n}(x) \\ &+a(l+m)\phi''(x)\phi^{m+2n+1}(x) + b_1\phi^2(x) \\ &+b_2\phi^{n+2}(x) + b_3\phi^{3n+2}(x) + b_4\phi^{4n+2}(x) \\ &-\lambda l\phi^{2n+2}(x) = 0. \end{aligned} \tag{18}$$

Similarly, implementing conditions as given by (5) equation (17) relaxes to

$$i(q^l)_t + a(|q|^m q^l)_{xx} + \left(\frac{b_1}{|q|^{2n}} + b_4|q|^{2n}\right) q^l = 0, \tag{19}$$

which in turn would reduce (18) to

$$\begin{aligned} &a(l+m)(l+m-1)\{\phi'(x)\}^2\phi^{m+2n}(x) \\ &+a(l+m)\phi''(x)\phi^{m+2n+1}(x) + b_1\phi^2(x) \\ &+b_4\phi^{4n+2}(x) - \lambda l\phi^{2n+2}(x) = 0. \end{aligned} \tag{20}$$

Just as before, Equation (20) permits a single Lie point symmetry, namely $\partial/\partial x$, which will assist in performing its integration. Therefore, integrating and discarding the integration constant, one arrives at the same solution given by (8) where, in this case:

$$B_1 = \frac{2b_4(2l+m)(2l+m-2n)\phi^{2n}}{\lambda l(2l+m-2n)(2l+m+2n)-P}, \tag{21}$$

with

$$P = \sqrt{\begin{aligned} &(2l+m-2n)(2l+m+2n) \\ &\times\begin{Bmatrix} \lambda^2 l^2(2l+m-2n)(2l+m+2n) \\ -4b_1b_4(2l+m)^2 \end{Bmatrix} \end{aligned}}, \tag{22}$$

and

$$B_2 = \frac{2b_4(2l+m)(2l+m-2n)\phi^{2n}}{P+\lambda l(2l+m-2n)(2l+m+2n)}, \tag{23}$$

while

$$B_3 = -\frac{2a(l+m)(2l+m-2n)\phi^{m+2n}}{b_1(m+2n)^2}. \tag{24}$$

Then, the constraint condition (14), in this situation, would generalize to

$$\phi(x) < \left|\frac{\lambda l(2l+m-2n)(2l+m+2n)-P}{2b_4(2l+m)(2l+m-2n)}\right|^{\frac{1}{2n}}. \tag{25}$$

The constraint condition (16) would still remain valid for generalized temporal evolution. It must be noted that the results of this subsection would fall back to the results of the previous subsection upon setting $l = -1$.

3. Conclusions

This paper recovered stationary optical soliton solutions to the Kudryashov's model having nonlinear CD. Both, linear temporal evolution as well as generalized temporal evolution are considered. In either case, the results are recovered in terms of Appell's hypergeometric function. The results show that CD, an important component of soliton formation, cannot be rendered to be nonlinear. This would stall soliton transmission through optical fibers across intercontinental distances. Therefore, it is imperative to make sure that CD stays linear throughout the fiber length. The results of this work would later be extended to additional models that govern the propagation dynamics of solitons through optical fibers and other form of waveguides.

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