Singular optical solitons with quadratic nonlinearity

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In this paper, we established a traveling wave solution by using sech-csch function algorithm for quadratic law medium in presence of spatio-temporal dispersion and intermodal dispersion. Singular solitons are thus obtained and listed.

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1. Introduction

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical physics and geochemistry [1-18]. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, tanh-coth method [8, 9, 14], extended tanh method [4, 5, 15], hyperbolic function method [16, 17], Jacobi elliptic function expansion method [7], F-expansion method [18], and the First Integral method [3, 6]. The sine-cosine method [8, 10] has been used to solve different types of nonlinear systems of PDEs.

This paper studies one such nonlinear evolution equation that appears in nonlinear optics. It is the nonlinear Schrodinger's equation with quadratic law nonlinearity. The search will be for singular soliton solution using sech-csch function method. In the past, singular solitons were retrieved for quadratic law medium by the aid of Q-function method, Riccati equation expansion scheme, G'/G-expansion, mapping method and the method of undetermined coefficients that is also known as ansatz approach [1, 11-13]. This paper is exclusively devoted to the retrieval of singular solitons to quadratic nonlinear medium using the sech-csch algorithm. The following section gives a quick review of this integration scheme.

2. The sech-csch function method

Consider the nonlinear partial differential equation in the form

$$F\left(u,u_t,u_x,u_y,u_{tt},u_{xx},u_{xy},u_{yy},.........\right)=0 \quad (1)$$

where u(x, y, t) is a traveling wave solution of nonlinear partial differential equation Eq. (1). We use the transformations,

$$u\left(x,y,t\right)=f\left(\xi\right) \quad (2)$$

where $\xi = x + y - \lambda t.$ This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.)=-\lambda\frac{d}{d\xi}(.),\frac{\partial}{\partial x}(.)=\frac{d}{d\xi}(.),\frac{\partial}{\partial y}(.)=\frac{d}{d\xi}(.) \quad (3)$$

Using Eq. (3) to transfer the nonlinear partial differential equation Eq. (1) to nonlinear ordinary differential equation

$$Q\left(f,f',f'',f''',\cdots\cdots\right)=0 \quad (4)$$

The ordinary differential equation (4) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form: [Anwar 2013]

$$\left.\begin{array}{l} f\left(\xi\right)=\alpha\,\mathrm{sec}\,h^{\beta}\left(\mu\xi\right) \\ \text{or in the form} \\ f\left(\xi\right)=\alpha\,\mathrm{csc}\,h^{\beta}\left(\mu\xi\right) \end{array}\right\} \quad (5)$$

where $\alpha, \mu,$ and β are parameters to be determined, μ and c are the wave number and the wave speed, respectively. We use

$$\left.\begin{array}{l} f(\xi)=\alpha \sec h^{\beta}(\mu\xi) \\ f^{'}(\xi)=-\alpha\,\beta\,\mu\,\sec h^{\beta}(\mu\xi).\tanh(\mu\xi) \\ f^{"}(\xi)=-\alpha\,\beta\,\mu^{2}\begin{bmatrix}(\beta+1)\sec h^{\beta+2}(\mu\xi)\\ -\beta\sec h^{\beta}(\mu\xi)\end{bmatrix} \\ f^{"'}(\xi)=\alpha\,\beta\,\mu^{3}\begin{bmatrix}(\beta+1)(\beta+2)\sec h^{\beta+2}(\mu\xi)\\ -\beta^{2}\sec h^{\beta}(\mu\xi)\end{bmatrix} \\ \tanh(\mu\xi) \end{array}\right\} \quad (6)$$

and their derivative. Or use

$$\left.\begin{array}{l} f(\xi)=\alpha \csc h^{\beta}(\mu\xi) \\ f^{'}(\xi)=-\alpha\,\beta\,\mu\,\csc h^{\beta}(\mu\xi).\coth(\mu\xi) \\ f^{"}(\xi)=\alpha\,\beta\,\mu^{2}\begin{bmatrix}(\beta+1)\csc h^{\beta+2}(\mu\xi)\\ +\beta\csc h^{\beta}(\mu\xi)\end{bmatrix} \\ f^{"'}(\xi)=-\alpha\,\beta\,\mu^{3}\begin{bmatrix}(\beta+1)(\beta+2)\csc h^{\beta+2}(\mu\xi)\\ +\beta^{2}\sec h^{\beta}(\mu\xi)\end{bmatrix} \\ \coth(\mu\xi) \end{array}\right\} \quad (7)$$

and so on. We substitute (6) or (7) into the reduced equation (4), balance the terms of the sech functions when (6) are used, or balance the terms of the csch functions when (7) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\sec h^{k}(\mu\xi)$ or $\csc h^{k}(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's $\alpha,\mu,$ and β, and solve the subsequent system.

3. Applications

For quadratic nonlinear media, with inter-modal dispersion (IMD) and spatio-temporal dispersion (STD) is given by

$$i\,q_t+a_1\,q_{xx}+b_1\,q_{xt}+c_1\,q+k_1\,q.r=i\,\lambda_1\,q_x \quad (8)$$

$$i\,r_t+a_2\,r_{xx}+b_2\,r_{xt}+c_2\,r+k_2\,q^2=i\,\lambda_2\,r_x \quad (9)$$

This equation studied by [SACHIN et al 2015].

where, $q(x,t)$ and $r(x,t)$ represents the wave profile of the fundamental harmonic and second harmonic components respectively and q^* represents conjugate of q. Also $a_j(j=1,2)$ are the coefficients of group velocity dispersion and b_j are the coefficients of STD while the right hand sides of (8) and (9) are IMD terms.

Introduce the transformations

$$\begin{array}{ll} q(x,t)=e^{i\theta(x,t)}.u(\xi), & r(x,t)=e^{i2\theta(x,t)}.\mathrm{v}(\xi) \\ \theta=-\tau\,x+\omega t+\in_0, & \xi=k_0\,(x-2\,\tau\,t+\chi) \end{array} \quad (10)$$

where $\tau,\omega,\in_0,k_0$, and χ are real constants. Substituting (10) into Equations (8-9) we obtain that

$$A_1\,.u(\xi)+B_1\,.u^{"}(\xi)+k_1\,.u(\xi).v(\xi)=0 \quad (11)$$

$$A_2\,.\mathrm{v}(\xi)+B_2\,.\mathrm{v}^{"}(\xi)+k_2\,.u^2(\xi)=0 \quad (12)$$

where

$$A_1=\left[c_1+\tau\,b_1\,\omega-\tau\,\lambda_1-a_1\,\tau^2-\omega\right] \quad (13)$$

$$B_1=\left[a_1-2\,\tau\,b_1\,\right]k_0^2 \quad (14)$$

$$A_2=2\left[\frac{c_2}{2}+2\tau\,b_2\,\omega-\tau\,\lambda_2-2a_2\,\tau^2-\omega\right] \quad (15)$$

$$B_2=\left[a_2-2\,\tau\,b_2\,\right]k_0^2 \quad (16)$$

and the relationships:

$$2\,\omega b_2-2\,\tau\left[1+2\,a_2-2\,\tau\,b_2\right]-\lambda_2=0 \quad (17)$$

$$b_1\omega-2\,\tau\left[1+a_{1.}-b_1\,\tau\,\right]-\lambda_1=0 \quad (18)$$

Seeking the solution by sech function method as in (6)

$$u(\xi)=\sigma_1\csc h^{\beta_2}(\mu\xi) \quad (19)$$

$$v(\xi)=\sigma_2\csc h^{\beta_2}(\mu\xi) \quad (20)$$

The the system of equations in Eqs. (11) and (12) becomes respectively:

$$\begin{array}{l} A_1\,\sigma_1\csc h^{\beta_1}(\mu\xi)+\sigma_1\,\beta_1\,\mu^2\,B_1\begin{bmatrix}(\beta_1+1)\\ \csc h^{\beta_1+2}(\mu\xi)\end{bmatrix} \\ +k_1\,\sigma_1\,\sigma_2\csc h^{\beta_1+\beta_2}(\mu\xi)=0 \end{array} \quad (21)$$

$$\begin{array}{l} A_2\,\sigma_2\csc h^{\beta_2}(\mu\xi)+\sigma_2\,\beta_2\,\mu^2\,B_2 \\ \begin{bmatrix}(\beta_2+1)\\ \csc h^{\beta_2+2}(\mu\xi)+\beta_2\csc h^{\beta_2}(\mu\xi)\end{bmatrix} \\ +k_2\,\sigma_1^2\csc h^{2\beta_1}(\mu\xi)=0 \end{array} \quad (22)$$

Equating the exponents and the coefficients of each pair of the csch functions we find the following algebraic system

$$\begin{aligned} 2\beta_1 &= \beta_2 + 2 \\ \beta_1 + \beta_2 &= \beta_1 + 2 \end{aligned} \tag{23}$$

Then

$$\beta_1 = 2, \quad \beta_2 = 2 \tag{24}$$

Thus setting coefficients of Equations (21-22) to zero yields

$$A_1\,\sigma_1 \operatorname{csc}h^2(\mu\xi) + 2\sigma_1\ \mu^2 B_1 \begin{bmatrix} 3\operatorname{csc}h^4(\mu\xi) + \\ 2\operatorname{csc}h^2(\mu\xi) \end{bmatrix} + k_1\,\sigma_1\,\sigma_2 \operatorname{csc}h^4(\mu\xi) = 0 \tag{25}$$

$$A_2\,\sigma_2 \operatorname{csc}h^2(\mu\xi) + 2\sigma_2\ \mu^2 B_1 \begin{bmatrix} 3\operatorname{csc}h^4(\mu\xi) + \\ 2\operatorname{csc}h^2(\mu\xi) \end{bmatrix} + k_2\,\sigma_1^2 \operatorname{csc}h^4(\mu\xi) = 0 \tag{26}$$

$$\left.\begin{aligned} A_1\,\sigma_1 + 4\sigma_1\,\mu^2\,B_1 &= 0 \\ 6\sigma_1\,\mu^2\,B_1 + k_1\,\sigma_1\,\sigma_2 &= 0 \\ A_2\,\sigma_2 + 4\sigma_2\,\mu^2\,B_2 &= 0 \\ 6\sigma_2\,\mu^2\,B_2 + k_2\,\sigma_1^2 &= 0 \end{aligned}\right\} \tag{27}$$

Solving the system of equations in (27) we get:

$$\mu^2 = -\frac{A_1}{4B_1} = -\frac{A_2}{4B_2}$$

Then

$$\frac{A_1}{B_2} = \frac{A_2}{B_2} \tag{28}$$

$$\sigma_2 = \frac{3A_1}{2k_1}, \quad \sigma_1 = \pm\frac{3}{2}\sqrt{\frac{A_1 A_2}{k_1 k_2}}, \tag{29}$$

Now from (28)

$$\frac{\left[c_1 + \tau\, b_1\,\omega - \tau\,\lambda_1 - a_1\,\tau^2 - \omega\right]}{\left[a_1 - 2\tau\, b_1\right]} = \frac{2\left[\frac{1}{2}c_2 + 2\tau\,\omega\, b_2 - \tau\,\lambda_2 - 2a_2\,\tau^2 - \omega\right]}{\left[a_2 - 2\tau\, b_2\right]}$$

we have following restrictions on coefficients:

$$\lambda_1 = \lambda_2 = \lambda \tag{30}$$

$$a_1 = 2a_2 = a \tag{31}$$

$$b_1 = 2b_2 = b \tag{32}$$

$$c_2 = 2c_1 = c \tag{33}$$

$$\tau \neq \frac{a}{2b} \tag{34}$$

Then:

$$A_1 = \frac{1}{2}A_2 = A = \left[\frac{c}{2} - \tau\,\lambda - a\,\tau^2 + (\tau\, b - 1)\,\omega\right] \tag{35}$$

$$B_1 = 2\,B_2 = B = \left[a - 2\,\tau\, b\right] k_0^2 \tag{36}$$

And from (17) and (18) we get:

$$\omega = \frac{2\,\tau\left[1 + a - \tau\, b\right] + \lambda}{b} \tag{37}$$

$$\mu = \frac{1}{2\,k_0}\sqrt{\frac{\frac{c}{2} - \tau\,\lambda - a\,\tau^2 + (\tau\, b - 1)\,\omega}{\left[2\,\tau\, b - a\right]}} \tag{38}$$

$$\sigma_1 = \pm 3\left[\frac{c}{2} - \tau\,\lambda - a\,\tau^2 + (\tau\, b - 1)\,\omega\right]\sqrt{\frac{1}{2\,k_1\,k_2}} \tag{39}$$

$$\sigma_1 = -\frac{3}{2\,k_1}\left[\frac{c}{2} - \tau\,\lambda - a\,\tau^2 + (\tau\, b - 1)\,\omega\right] \tag{40}$$

Then:

$$u(x,t) = \pm 3\left[\frac{c}{2} - \tau\,\lambda - a\,\tau^2 + (\tau\, b - 1)\,\omega\right] \sqrt{\frac{1}{2\,k_1\,k_2}} \operatorname{csc}h^2\left(\frac{1}{2}\sqrt{\frac{\left[\frac{c}{2} + \tau\, b\,\omega - \tau\,\lambda - a\,\tau^2 - \omega\right]}{\left[2\,\tau\, b - a\right]}}\,(x - 2\,\tau\, t + \chi)\right) \tag{41}$$

$$v(x,t) = -\frac{3}{2\,k_1}\left[\frac{c}{2} - \tau\,\lambda - a\,\tau^2 + (\tau\, b - 1)\,\omega\right] \operatorname{csc}h^2 \sqrt{\frac{1}{2\,k_1\,k_2}} \operatorname{csc}h^2\left(\frac{1}{2}\sqrt{\frac{\left[\frac{c}{2} + \tau\, b\,\omega - \tau\,\lambda - a\,\tau^2 - \omega\right]}{\left[2\,\tau\, b - a\right]}}\,(x - 2\,\tau\, t + \chi)\right) \tag{42}$$

$$q(x,\mathrm{t}) = 3e^{i\theta(x,t)}\left[\frac{c}{2}-\tau\lambda-a\tau^2+(\tau b-1)\omega\right]$$
$$\sqrt{\frac{1}{2k_1k_2}}\operatorname{csc}h^2\left(\frac{1}{2}\sqrt{\frac{\left[\frac{c}{2}+\tau b\omega-\tau\lambda-a\tau^2-\omega\right]}{[2\tau b-a]}}(x-2\tau t+\chi)\right) \tag{43}$$

$$r(x,\mathrm{t}) =$$
$$-\frac{3}{2k_1}e^{i2\theta(x,t)}\left[\frac{c}{2}-\tau\lambda-a\tau^2+(\tau b-1)\omega\right]\operatorname{csc}h^2 \tag{44}$$
$$\sqrt{\frac{1}{2k_1k_2}}\operatorname{csc}h^2\left(\frac{1}{2}\sqrt{\frac{\left[\frac{c}{2}+\tau b\omega-\tau\lambda-a\tau^2-\omega\right]}{[2\tau b-a]}}(x-2\tau t+\chi)\right)$$

where:

$$\theta = -\tau x + \left(\frac{2\tau[1+a-\tau b]+\lambda}{b}\right)t + \in_0 \tag{45}$$

4. Conclusions

In this paper, the sech-csch function method has been successfully applied to find singular solitons for quadratic law nonlinear medium. This is one of the very many integration algorithms to locate singular soliton solutions. Later this scheme will be applied to other scenarios such as birefringent fibers, DWDM systems, optical metamaterials and others. Such results will be reported soon.

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