Optical solitons with quadratic nonlinearity by Lie symmetry analysis

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This paper studies similarity solutions of quadratic nonlinear media in presence of spatio-temporal dispersion and inter-modal dispersion. Lie classical method is applied to quadratic nonlinear media and some exact solutions are presented. Later singular soliton solutions are also retrieved by the ansatz approach.

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1. Introduction

Optical solitons is one of the most fascinating areas of research in nonlinear fibers optics. There are several kinds of solitons that are retrieved from the model that governs this study. These are bright, dark and singular solitons. Several results are reported in this context. Additionally several integration techniques are applied to retrieve these results. These include Lie symmetry, G'/G-expansion, traveling wave hypothesis, semi-inverse variational principle and several others. Therefore, changing gears, this paper focuses on solitons with quadratic nonlinear medium. While a plethora of results already exists, this paper focuses on this topic by Lie symmetry analysis for the first time [1-20].

The nonlinear effect in quadratic media is the form of second harmonic generation (SHG). The pump wave at the fundamental harmonic (FH) generates a second harmonic (SH) with double frequency [1, 15]. This SHG phenomena is derivable from Maxwell's equation with quadratic nonlinear media. The solitons in quadratic nonlinear media are studied in several areas of nonlinear optics such as optical routing, optical switching, lasers with quadratic nonlinear crystal and others [1, 15].

2. Governing equations

For quadratic nonlinear media, with inter-modal dispersion (IMD) and spatio-temporal dispersion (STD) is given by

$$\begin{aligned} &iq_t + a_1 q_{xx} + b_1 q_{xt} \\ &+ c_1 q + k_1 q^* r = i\alpha_1 q_x \end{aligned} \tag{1-1}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 r + k_2 q^2 = i\alpha_2 r_x \tag{1-2}$$

where, $q(x,t)$ and $r(x,t)$ represents the wave profile of the FH and SH components respectively. q^* represents conjugate of q. The independent variables are x and t that are spatial and temporal variables. The inclusion of STD, given by the coefficients of b_j for $j=1,2,$ was suggested during 2012 [16, 17]. With inclusion of the STD, the governing NLSE becomes well-posed as opposed to the consideration of group velocity dispersion (GVD) alone, in which case, the model problem stays ill-posed [16, 17]. Recently soliton solutions were obtained for quadratic nonlinear media in presence of GVD only [1] and with IMD and STD [15].

3. Admissible transformations

To separate the real and imaginary parts, let us assume

$$q(x,t) = P_1(x,t)e^{i\phi(x,t)} \tag{2-1}$$

$$r(x,t) = P_2(x,t)e^{2i\phi(x,t)} \tag{2-2}$$

where

$$\phi(x,t) = -\kappa x + \omega t + \theta \tag{3}$$

Here κ is the frequency of the wave and ω is the wave number of the soliton. Also, θ is the phase constant. Substituting (2) into the system of equations (1) and decomposing into real and imaginary parts gives

$$P_1(\omega + a_1\kappa^2 - b_1\kappa\omega + \kappa\alpha_1 - c_1) - a_1\frac{\partial^2 P_1}{\partial x^2} - b_1\frac{\partial^2 P_1}{\partial x \partial t} - k_1 P_1 P_2 = 0 \quad (4\text{-}1)$$

$$(b_1\kappa - 1)\frac{\partial P_1}{\partial t} - (2a_1\kappa - b_1\omega + \alpha_1)\frac{\partial P_1}{\partial x} = 0 \quad (4\text{-}2)$$

$$P_2(2\omega + 4a_2\kappa^2 - 4b_2\kappa\omega + 2\kappa\alpha_2 - c_2) - a_2\frac{\partial^2 P_2}{\partial x^2} - b_2\frac{\partial^2 P_2}{\partial x \partial t} - k_2 P_1^2 = 0 \quad (4\text{-}3)$$

$$(2b_2\kappa - 1)\frac{\partial P_2}{\partial t} - (4a_2\kappa - 2b_2\omega + \alpha_2)\frac{\partial P_2}{\partial x} = 0 \quad (4\text{-}4)$$

For eliminating the imaginary parts let us assume following constraints

$$\kappa = \frac{1}{b_1} = \frac{1}{2b_2} \quad (5\text{-}1)$$

$$\omega = \frac{2a_1k + \alpha_1}{b_1} = \frac{4a_2k + \alpha_2}{2b_2} \quad (5\text{-}2)$$

From (5), we have following restrictions on coefficients

$$a_1 = 2a_2 = 2a \quad (6\text{-}1)$$

$$b_1 = 2b_2 = 2b \quad (6\text{-}2)$$

$$\alpha_1 = \alpha_2 = \alpha \quad (6\text{-}3)$$

The detail are analyzed from equation (29) in Section-4. With conditions (5) and (6), system of equations (2) reduces to

$$P_1(\omega + 2a\kappa^2 - 2b\kappa\omega + \kappa\alpha - c_1) - 2a\frac{\partial^2 P_1}{\partial x^2} - 2b\frac{\partial^2 P_1}{\partial x \partial t} - k_1 P_1 P_2 = 0 \quad (7\text{-}1)$$

$$P_2(2\omega + 4a\kappa^2 - 4b\kappa\omega + 2\kappa\alpha - c_2) - a\frac{\partial^2 P_2}{\partial x^2} - b\frac{\partial^2 P_2}{\partial x \partial t} - k_2 P_1^2 = 0 \quad (7\text{-}2)$$

Now we will apply Lie classical method to the system (7). Roughly speaking an admissible transformation is a triple consisting of two fixed equations from a class and a point transformation linking these equations. The set of admissible transformations of a class of differential equations possesses the groupoid structure with respect to the standard composition of transformations [3, 14]. Therefore, we look for Lie symmetry operators of the form

$$Q = \tau(t,x,P_1,P_2)\partial_t + \xi(t,x,P_1,P_2)\partial_x + \eta_1(t,x,P_1,P_2)\partial_{p_1} + \eta_2(t,x,P_1,P_2)\partial_{p_2} \quad (8)$$

where τ, ξ, η_1 and η_2 are infinitesimals that generate one-parametric Lie groups of transformations leaving equations from class (7) invariant.

Applying the second prolongation $pr^{(2)}Q$ to system (7), we find that the coefficient functions $\tau(t,x,P_1,P_2)$, $\xi(t,x,P_1,P_2)$, $\eta_1(t,x,P_1,P_2)$ and $\eta_2(t,x,P_1,P_2)$ must satisfy the symmetry condition

$$\eta_1(\omega + 2a\kappa^2 - 2b\kappa\omega + \kappa\alpha - c_1) - 2a\eta_1^{xx} - 2b\eta_1^{xt} - k_1(\eta_1 P_2 + \eta_2 P_1) = 0 \quad (9\text{-}1)$$

$$\eta_2(2\omega + 4a\kappa^2 - 4b\kappa\omega + 2\kappa\alpha - c_2) - a\eta_2^{xx} - b\eta_2^{xt} - 2k_2\eta_1 P_1 = 0 \quad (9\text{-}2)$$

where η_1^{xx}, η_1^{xt}, η_2^{xx}, η_2^{xt} are infinitesimals [3, 14].

Substituting the value of η_1^{xx}, η_1^{xt}, η_2^{xx}, η_2^{xt}, and equating coefficients of derivative terms and powers of P_1, P_2, we obtain

$$\xi = C_2 + C_3(-x + \frac{2at}{b}) \quad (10\text{-}1)$$

$$\tau = C_1 + tC_3 \quad (10\text{-}2)$$

$$\eta_1 = 0 \quad (10\text{-}3)$$

$$\eta_2 = 0 \quad (10\text{-}4)$$

where C_1, C_2 and C_3 are arbitrary constants.

Corresponding vector fields are

$$V_1 = \frac{\partial}{\partial t} \quad (11\text{-}1)$$

$$V_2 = \frac{\partial}{\partial x} \quad (11\text{-}2)$$

$$V_3 = t\frac{\partial}{\partial t} + (-x + \frac{2at}{b})\frac{\partial}{\partial t} \qquad (11\text{-}3)$$

We will consider following vector fields for reduction of system (7)

(i) V_3

(ii) $V_1 + \mu V_2$

where μ is real arbitrary constant.

3. Reduced systems and exact solutions

3.1 CASE-I (Vector field: V_3)

The similarity variable and the form of similarity solution is as follows:

$$\sigma = xt - \frac{a}{b}t^2 \qquad (12\text{-}1)$$

$$P_1 = F(\sigma) \qquad (12\text{-}2)$$

$$P_2 = G(\sigma) \qquad (12\text{-}3)$$

where σ is new independent variable and F, G are new dependent variables.

Substituting (12) in system of equations (7), we obtain following ordinary differential equations (ODEs)

$$\begin{aligned}&(\omega + 2a\kappa^2 - 2b\kappa\omega + \kappa\alpha - c_1)F \\ &- 2a\sigma F'' - 2bF' - k_1 FG = 0\end{aligned} \qquad (13\text{-}1)$$

$$\begin{aligned}&(2\omega + 4a\kappa^2 - 4b\kappa\omega + 2\kappa\alpha - c_2)G \\ &- a\sigma G'' - bG' - k_2 F^2 = 0\end{aligned} \qquad (13\text{-}2)$$

where ’ denotes derivative with respect to σ.

This ODE system can be further solved to give solution

$$F(\sigma) = -\sqrt{\frac{2}{k_1 k_2}}\frac{b}{\sigma} \qquad (14\text{-}1)$$

$$G(\sigma) = -\frac{2b}{k_1\sigma} + \frac{(-2c_1 b^2 + b\alpha + a)}{2k_1 b^2} \qquad (14\text{-}2)$$

Corresponding solutions of main system (1) is given by

$$q(x,t) = -\sqrt{\frac{2}{k_1 k_2}}\frac{b}{(xt - \frac{a}{b}t^2)}e^{i(-\kappa x + \omega t + \theta)} \qquad (15\text{-}1)$$

$$r(x,t) = -\left\{\begin{array}{l}\dfrac{2b}{k_1(xt - \frac{a}{b}t^2)} \\ + \dfrac{(-2c_1 b^2 + b\alpha + a)}{2k_1 b^2}\end{array}\right\}e^{2i(-\kappa x + \omega t + \theta)} \qquad (15\text{-}2)$$

where κ and ω are given by (5) with (6).

3.2 CASE-II (Vector field: $V_1 + \mu V_2$)

Similarity variables are as follows

$$\sigma = x - \mu t \qquad (16\text{-}1)$$

$$P_1 = F(\sigma) \qquad (16\text{-}2)$$

$$P_2 = G(\sigma) \qquad (16\text{-}3)$$

where σ is new independent variable and F, G are new dependent variables.

Substituting in (7), we obtain following system of ODEs

$$\begin{aligned}&-(2a + 2b\mu)F'' - k_1 FG \\ &+ (\omega + 2a\kappa^2 - 2b\kappa\omega + \kappa\alpha - c_1)F = 0\end{aligned} \qquad (17\text{-}1)$$

$$\begin{aligned}&-(a - b\mu)G'' - k_2 F^2 \\ &+ (2\omega + 4a\kappa^2 - 4b\kappa\omega + 2\kappa\alpha - c_2)G = 0\end{aligned} \qquad (17\text{-}2)$$

where ’ denotes derivative with respect to σ.

Employing the conditions

$$c_2 = \frac{a + \alpha b}{b^2} \qquad (18)$$

we obtain the following solution of (17)

$$F(\sigma) = \pm 6\sqrt{\frac{2}{k_1 k_2}}\frac{(a - b\mu)}{\sigma^2} \qquad (19\text{-}1)$$

$$G(\sigma) = -\frac{12(a - b\mu)}{k_1\sigma^2} + \frac{a + \alpha b - 2c_1 b^2}{2k_1 b^2} \qquad (19\text{-}2)$$

Corresponding solutions of main system (1) with (18), is given by

$$q(x,t)=\left\{\pm 6\sqrt{\frac{2}{k_1 k_2}}\frac{(a-b\mu)}{(x-\mu t)^2}\right\} \times e^{i(-\kappa x+\omega t+\theta)} \tag{20-1}$$

$$r(x,t)=\left\{-\frac{12(a-b\mu)}{k_1(x-\mu t)^2}+\frac{a+\alpha b-2c_1 b^2}{2k_1 b^2}\right\} \times e^{2i(-\kappa x+\omega t+\theta)} \tag{20-2}$$

where κ and ω are given by (5) with (6).

4. Singular solitons

This section will retrieve exact singular 1-soliton solutions to the model (1). It must be noted that singular solitons of the first type was reported during 2014 [15]. This paper will now report singular solitons of the second type. With the transformation (6), in place, the model, given by (6) reduces to

$$iq_t + 2aq_{xx} + 2bq_{xt} + c_1 q + k_1 q^* r = i\alpha q_x \tag{21}$$

$$ir_t + ar_{xx} + br_{xt} + c_2 r + k_2 q^2 = i\alpha r_x \tag{22}$$

This pair of equations (21) and (22) will now be solved for singular 1-soliton solution. The starting hypothesis is [15]:

$$P_l(x,t) = A_l + B_l \coth^{p_l}\tau \tag{23}$$

for $l = 1, 2$. Here A_l, B_l and B are free parameters where τ is given by

$$\tau = B(x - vt) \tag{24}$$

Substituting the hypothesis (23) into (21) and (22) the real part relations are

$$\begin{aligned} &2p_1(p_1+1)(a-bv)B_1B^2\coth^{p_1+2}\tau \\ &-\left\{\begin{aligned}&B_1(\omega+2a\kappa^2-2b\kappa\omega+\kappa\alpha-c_1)\\&+4(a-bv)p_1^2B_1B^2-k_1B_1A_2\end{aligned}\right\} \\ &\times\coth^{p_1}\tau+2p_1(p_1-1)(a-bv)B_1B^2\coth^{p_1-2}\tau \\ &+k_1A_1B^2\coth^{p_2}\tau+k_1B_1B_2\coth^{p_1+p_2}\tau \\ &-A_1(\omega+2a\kappa^2-2b\kappa\omega+\kappa\alpha-c_1-k_1A_2)=0 \end{aligned} \tag{25}$$

and

$$\begin{aligned} &p_2(p_2+1)(a-bv)B_2B^2\coth^{p_2+2}\tau \\ &-\left\{\begin{aligned}&B_2(2\omega+4a\kappa^2-4b\kappa\omega+2\kappa\alpha-c_2)\\&+2(a-bv)p_2^2B_2B^2\end{aligned}\right\} \\ &\times\coth^{p_2}\tau+p_2(p_2-1)(a-bv)B_2B^2\coth^{p_2-2}\tau \\ &+k_2B_1^2\coth^{2p_1}\tau+k_2B_1^2\coth^{2p_1}\tau \\ &-A_2(2\omega+4a\kappa^2-4b\kappa\omega+2\kappa\alpha-c_2-k_2A_1^2)=0 \end{aligned} \tag{26}$$

respectively. Next, the imaginary parts give the speed of the solitons:

$$v=\frac{2a_1\kappa-b_1\omega+\alpha_1}{b_1\kappa-1} \tag{27}$$

and

$$v=\frac{4a_2\kappa-2b_2\omega+\alpha_2}{2b_2\kappa-1} \tag{28}$$

respectively from the two components. Equating the speed of the solitons of the two components from (27) and (28) gives

$$\begin{aligned} &4\kappa^2(a_1b_2-a_2b_1) \\ &+\kappa\{2\alpha_1b_2-\alpha_2b_1-2(a_1-2a_2)\} \\ &+\omega(b_1-2b_2)+(\alpha_2-\alpha_1)=0 \end{aligned} \tag{29}$$

It is this equation (29) that leads to the conclusions given by (6). Balancing principle applied to (25) and (26) yields

$$p_l = 2 \tag{30}$$

for $l = 1, 2$. The coefficients of linearly independent functions yield

$$v=\frac{12aB^2+k_1B_2}{12bB^2} \tag{31}$$

and

$$\omega=\frac{\left\{\begin{aligned}&3B_1(2a\kappa^2-c_1+\alpha\kappa)\\&-k_1(4B_1B_2+3B_1A_2+3A_1B_2)\end{aligned}\right\}}{3B_1(2b\kappa-1)} \tag{32}$$

from the first component (25). The coefficients of linearly independent functions from second component (26) gives

$$v=\frac{6aB_2B^2+k_2B_1^2}{6bB_2B^2} \tag{33}$$

and

$$\omega = \frac{3B_2(4a\kappa^2 - c_2 + \alpha\kappa) - 2k_2B_1(2B_1 + 3A_1)}{6B_2(2b\kappa - 1)} \quad (34)$$

Next equating the speed of singular solitons from (31) and (33) leads to the ratio of free parameters B_l for $l = 1, 2$ as

$$\frac{B_1}{B_2} = \sqrt{\frac{k_1}{2k_2}} \quad (35)$$

which again prompts the constraint given by

$$k_1k_2 > 0 \quad (36)$$

Next, setting the wave numbers, from (32) and (34), equal to one another yields the algebraic relation between the free parameters as

$$\begin{aligned}&3B_1B_2(2c_1 - c_2 + 2k_1A_2)\\ &-4k_2B_1^3 - 6A_1k_2B_1^2\\ &+8k_1B_1B_2^2 + 6kA_1B_2^2 = 0\end{aligned} \quad (37)$$

Now, equating the speed of the soliton from (27) and (31) leads to the free parameter B as

$$B = \frac{1}{2}\left[\frac{(2b\kappa - 1)k_1B_2}{3(2ab\kappa - 2b^2\omega + b\alpha + a)}\right]^{\frac{1}{2}} \quad (38)$$

that introduces the constraint condition

$$\begin{aligned}&k_1B_2(2b\kappa - 1)\\ &\times(2ab\kappa - 2b^2\omega + b\alpha + a) > 0\end{aligned} \quad (39)$$

Similary equating the speed of the soliton from (28) and (33) also yields (38) and (39).

Now, substituting the wave number ω from (32) into (38) gives

$$\begin{aligned}B = &\frac{2b\kappa - 1}{2}\\ &\times\sqrt{\frac{k_1B_1B_2}{\left\{\begin{array}{l}6b^2c_1B_1 + 2k_1b^2(4B_1B_2 + 3B_1A_2 + 3A_1B_2)\\ -3aB_1 - 3\alpha bB_1\end{array}\right\}}}\end{aligned} \quad (40)$$

with the constraint

$$k_1B_1B_2\left\{\begin{array}{l}6b^2c_1B_1 + 2k_1b^2(4B_1B_2 + 3B_1A_2 + 3A_1B_2)\\ -3aB_1 - 3\alpha bB_1\end{array}\right\} > 0 \quad (41)$$

From real part equations (25) and (26) subtracting the constant terms leads to

$$\begin{aligned}&2vbB^2(B_2 - 2B_1) + \omega(2b\kappa - 1)(A_1 - 2A_2)\\ &-2a\kappa^2(A_1 - 2A_2) - \alpha\kappa(A_1 - 2A_2)\\ &+\left\{\begin{array}{l}2aB^2(2B_1 - B_2) + c_1A_1\\ -c_2A_2 + k_1A_1A_2 - k_2A_1^2\end{array}\right\} = 0\end{aligned} \quad (42)$$

This leads to the conclusion, from the coefficients of independent parameters, and after implementing (41) and (42)

$$A_1 = 2A_2 = \frac{2c_1 - c_2}{3k} \quad (43)$$

and

$$B_1 = 2B_2 = \frac{3(c_2 - 2c_1)}{k} \quad (44)$$

where

$$k_2 = 2k_1 = 2k \quad (45)$$

Hence, for singular 1-soliton NLSE with quadratic nonlinearity, the model equations (21) and (22) further simplify to

$$iq_t + 2aq_{xx} + 2bq_{xt} + c_1q + kq^*r = i\alpha q_x \quad (46)$$

$$ir_t + ar_{xx} + br_{xt} + c_2r + 2kq^2 = i\alpha r_x \quad (47)$$

whose singular 1-soliton solution is:

$$\begin{aligned}q(x,t) = &(A_1 + B_1\coth^2\tau)\\ &\times e^{i(-\kappa x + \omega t + \theta)}\end{aligned} \quad (48)$$

$$\begin{aligned}r(x,t) = &(A_2 + B_2\coth^2\tau)\\ &\times e^{2i(-\kappa x + \omega t + \theta)}\end{aligned} \quad (49)$$

where the free parameters, speed and wave numbers are explicitly determined. The results of this section are eerie similar to the results for topological solitons that were reported during 2014 [15]. However, these are singular solitons while previously reported results were topological solitons. Therefore, mathematically speaking, the functions are different with the same structure of the results.

5. Conclusions

This paper studied optical solitons with quadratic law nonlinear medium. First Lie symmetry analysis retrieved a couple of solutions to the model. Later, ansatz approach obtained singular soliton solution to the model. It needs to be noted that this is a different kind of singular solitons as compared to the one that was reported earlier during 2014 [15]. The constraint conditions for the existence of these soliton solutions are also given. This problem will be studied in future with several additional integration techniques. These include Kudryashov's method, G'/G-expansion scheme and others. The results of these research will be reported soon.

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