Optical solitons of the KdV equation with power-law nonlinearity

YUSUF PANDIR[*]
Department of Mathematics, Faculty of Science and Arts, Bozok University, 66100 Yozgat, Turkey

This paper obtains solitons and elliptic *F*-function, elliptic *E*-function, elliptic Π-function solutions to the KdV equation with power-law nonlinearity. We consider and derive some new results using the so-called as the extended trial equation method. By means of this method, bright and dark optical solitons as well as the corresponding singular periodic solutions are obtained. This approach can also be applied to other nonlinear differential equations.

1. Introduction

The theoretical research on the physical system shows that optical solitons is still important. In order to obtain the optical soliton solutions, it should be examined the exact solution of nonlinear partial differential equations. There has been an increasing attention in finding the exact analytical solutions to nonlinear partial differential equations by using convenient techniques. Many different approaches have been presented in the literature to obtain optical soliton solutions such as the F-expansion method [1, 2], the modified simplest equation method [3, 4], the Lie group method [5], the homogeneous balance method [6], the sub-ODE method [7], (G'/G)-expansion method [8-10], the trial equation method [11-13], the linear superposition method [14, 15] and so on. Also, the authors in Refs. [16-22] introduced new versions of the trial equation method and so-called as extended trial equation method to search for exact solutions of the nonlinear partial differential equations. Soliton solutions, compactons, singular solitons, elliptic integral function solutions and Jacobi elliptic function solutions have been found by using these methods. These types of solutions appear in various areas of mathematical physics.

In Section 2, we implement a new trial equation method, which is firstly defined in the paper [18], for nonlinear evolution equations with higher order nonlinearity. In Section 3, as an application, we obtain bright optical soliton, dark optical soliton, singular soliton and Jacobi elliptic function solutions to the KdV equation with power-law nonlinearity [9, 23]

$$q_t + a(n+1)q^n q_x + b q_{xxx} = 0, \qquad (1)$$

where the coefficients a and b are not equal to zero, and $n > 2$. The KdV equation is very important problem in applied mathematics and physics. The KdV equation describes the evolution of one-dimensional waves in many physical settings, including shallow-water waves, long internal waves, ionacoustic waves, and more.

2. Method

The extended trial equation method

Step 1. We consider a nonlinear partial differential equation

$$P(u, u_t, u_x, u_{xx}, \cdots) = 0, \qquad (2)$$

and get the general traveling wave transformation

$$u(x_1, x_2, \cdots, x_N, t) = u(\eta), \qquad \eta = \lambda\left(\sum_{j=1}^{N} x_j - ct\right), \qquad (3)$$

where $\lambda \neq 0$ and $c \neq 0$. Substituting Eq. (3) into Eq. (2) yields a nonlinear ordinary differential equation

$$N(u, u', u'', u''', \cdots) = 0. \qquad (4)$$

Step 2. From Ref. [25], the general solutions of Eq. (4) are given as

$$u = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \qquad (5)$$

Where

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \cdots + \xi_1 \Gamma + \xi_0}{\zeta_\varepsilon \Gamma^\varepsilon + \cdots + \zeta_1 \Gamma + \zeta_0}. \qquad (6)$$

From Eqs. (5) and (6), we can write

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right)^2, \qquad (7)$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2} \right), \qquad (8)$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials. Substituting these relations into Eq. (4) yields an equation of polynomial $\Omega(\Gamma)$ of Γ:

$$\Omega(\Gamma) = \rho_s \Gamma^s + \cdots + \rho_1 \Gamma + \rho_0 = 0. \qquad (9)$$

According to the balance principle, we can find a relation of θ, ε and δ. We can compute some values of θ, ε and δ.

Step 3. Let the coefficients of $\Omega(\Gamma)$ all be zero will yield an algebraic equations system:

$$\rho_i = 0, \quad i = 0, \cdots, s. \qquad (10)$$

Solving this system, we will determine the values of $\xi_0, \cdots, \xi_\theta, \ \zeta_0, \cdots, \zeta_\varepsilon$ and $\tau_0, \cdots, \tau_\delta$.

Step 4. Reduce Eq. (6) to the elementary integral form

$$\pm(\eta - \eta_0) = \int \frac{d\Gamma}{\sqrt{\Lambda(\Gamma)}} = \int \sqrt{\frac{\Phi(\Gamma)}{\Psi(\Gamma)}} d\Gamma. \qquad (11)$$

Using the roots of $\Phi(\Gamma)$, we solve Eq. (11) with the help of MATHEMATICA and classify the exact solutions to Eq. (2).

Application and results

In this section, we will construct traveling wave solutions of the generalized forms of the KdV equations by using the extended trial equation method. In order to look for travelling wave solutions of Eq. (1), we make the transformation $q(x,t) = u(\eta), \quad \eta = x - ct$, where c is the wave speed. Therefore, it can be converted to the ordinary differential equation

$$-c\big(u(\eta)\big)' + a(n+1)u^n(\eta)\big(u(\eta)\big)' + b\big(u(\eta)\big)''' = 0, \qquad (12)$$

where the prime denotes the derivative with respect to η. Then, integrating this equation with respect to η one time and setting the integration constant to zero, we obtain

$$-cu(\eta) + au^{n+1}(\eta) + b\big(u(\eta)\big)'' = 0. \qquad (13)$$

Using the transformation

$$u = v^{\frac{1}{n}}, \qquad (14)$$

Eq. (13) turns into the equation

$$bnvv'' + b(1-n)(v')^2 - cn^2 v^2 + an^2 v^3 = 0. \qquad (15)$$

Substituting Eqs. (7) and (8) into Eq. (15) and using balance principle yields

$$\theta = \varepsilon + \delta + 2.$$

By taking into account balancing procedure, we get results as follows:

Case 1: If we take $\varepsilon = 0$, $\delta = 1$ and $\theta = 3$, then

$$(v')^2 = \frac{\tau_1^2 \left(\xi_3 \Gamma^3 + \xi_2 \Gamma^2 + \xi_1 \Gamma + \xi_0 \right)}{\zeta_0}, \quad v'' = \frac{\tau_1 \left(3\xi_3 \Gamma^2 + 2\xi_2 \Gamma + \xi_1 \right)}{2\zeta_0}, \qquad (16)$$

where $\xi_3 \neq 0, \zeta_0 \neq 0$. Thus, we have a system of algebraic equations from the coefficients of polynomial of Γ. Solving this algebraic system, we get

$$\xi_0 = -\frac{\tau_0 \left(\xi_2 \tau_0 - 2\xi_1 \tau_1 \right)}{3\tau_1^2}, \qquad \xi_3 = \frac{\tau_1 \left(2\xi_2 \tau_0 - \xi_1 \tau_1 \right)}{3\tau_0^2}, \qquad (17)$$

$$\zeta_0 = -\frac{b(n+2)\left(2\xi_2 \tau_0 - \xi_1 \tau_1 \right)}{6an^2 \tau_0^2}, \qquad c = \frac{6a\tau_0 \left(\xi_2 \tau_0 - \xi_1 \tau_1 \right)}{(n+2)\left(2\xi_2 \tau_0 - \xi_1 \tau_1 \right)}, \qquad (18)$$

where ξ_1, ξ_2, τ_0 and τ_1 are free parameters. Substituting these results into Eqs. (6) and (11), we have

$$\pm(\eta - \eta_0) = A \int \frac{d\Gamma}{\sqrt{\Gamma^3 + \dfrac{3\xi_2 \tau_0^2 \Gamma^2}{\tau_1 \left(2\xi_2 \tau_0 - \xi_1 \tau_1 \right)} + \dfrac{3\xi_1 \tau_0^2 \Gamma}{\tau_1 \left(2\xi_2 \tau_0 - \xi_1 \tau_1 \right)} - \dfrac{\tau_0^3 \left(\xi_2 \tau_0 - 2\xi_1 \tau_1 \right)}{\tau_1^3 \left(2\xi_2 \tau_0 - \xi_1 \tau_1 \right)}}}, \qquad (19)$$

where $A = \sqrt{-\dfrac{b(n+2)}{2an^2 \tau_1}}$. Integrating Eq. (19), we obtain the following solutions to Eq. (1). These solutions are respectively:

$$\pm(\eta - \eta_0) = -\frac{2A}{\sqrt{\Gamma - \alpha_1}}, \qquad (20)$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{\alpha_2 - \alpha_1}} \arctan\sqrt{\frac{\Gamma - \alpha_2}{\alpha_2 - \alpha_1}}, \quad \alpha_2 > \alpha_1, \quad (21)$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{\alpha_1 - \alpha_2}} \ln\left|\frac{\sqrt{\Gamma - \alpha_2} - \sqrt{\alpha_1 - \alpha_2}}{\sqrt{\Gamma - \alpha_2} + \sqrt{\alpha_1 - \alpha_2}}\right|, \quad \alpha_1 > \alpha_2, \quad (22)$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{\alpha_1 - \alpha_3}} F(\varphi, l), \quad \alpha_1 > \alpha_2 > \alpha_3, \quad (23)$$

where

$$F(\varphi, l) = \int_0^\varphi \frac{d\Psi}{\sqrt{1 - l^2 \sin^2 \Psi}}, \quad \varphi = Arc\sin\sqrt{\frac{(\Gamma - \alpha_3)}{(\alpha_2 - \alpha_3)}},$$

$$l^2 = \frac{(\alpha_2 - \alpha_3)}{(\alpha_1 - \alpha_3)}. \qquad (24)$$

Also α_1, α_2 and α_3 are the roots of the polynomial equation

$$\Gamma^3 + \frac{\xi_2}{\xi_3}\Gamma^2 + \frac{\xi_1}{\xi_3}\Gamma + \frac{\xi_0}{\xi_3}. \qquad (25)$$

Substituting the solutions (20)-(23) into (5) and (14), we have

$$q(x,t) = \left[\tau_0 + \tau_1\alpha_1 + \frac{4\tau_1 A^2}{\left(x - \frac{6a\tau_0(\xi_2\tau_0 - \xi_1\tau_1)}{(n+2)(2\xi_2\tau_0 - \xi_1\tau_1)}t - \eta_0\right)^2}\right]^{\frac{1}{n}}, \qquad (26)$$

$$q(x,t) = \left[\begin{array}{l}\tau_0 + \tau_1\alpha_1 \\ +\tau_1(\alpha_2 - \alpha_1)\operatorname{sech}^2\left(\frac{1}{2}\frac{\sqrt{\alpha_2 - \alpha_1}}{A}\left(x - \frac{6a\tau_0(\xi_2\tau_0 - \xi_1\tau_1)}{(n+2)(2\xi_2\tau_0 - \xi_1\tau_1)}t - \eta_0\right)\right)\end{array}\right]^{\frac{1}{n}}, \qquad (27)$$

$$q(x,t) = \left[\begin{array}{l}\tau_0 + \tau_1\alpha_1 \\ +\tau_1(\alpha_1 - \alpha_2)\cos ech^2\left(\frac{1}{2}\frac{\sqrt{\alpha_1 - \alpha_2}}{A}\left(x - \frac{6a\tau_0(\xi_2\tau_0 - \xi_1\tau_1)}{(n+2)(2\xi_2\tau_0 - \xi_1\tau_1)}t - \eta_0\right)\right)\end{array}\right]^{\frac{1}{n}} \qquad (28)$$

and

$$q(x,t) = \left[\begin{array}{l}\tau_0 + \tau_1\alpha_3 \\ +\tau_1(\alpha_2 - \alpha_3)sn^2\left(\mp\frac{1}{2}\frac{\sqrt{\alpha_1 - \alpha_3}}{A}\left(x - \frac{6a\tau_0(\xi_2\tau_0 - \xi_1\tau_1)}{(n+2)(2\xi_2\tau_0 - \xi_1\tau_1)}t - \eta_0\right), \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}\right)\end{array}\right]^{\frac{1}{n}}. \qquad (29)$$

If we take $\tau_0 = -\tau_1\alpha_1$ and $\eta_0 = 0$ for simplicity, then the solutions (26)-(28) can be reduced to rational function solution

$$q(x,t) = \left(\frac{\tilde{A}}{x - ct}\right)^{\frac{2}{n}}, \qquad (30)$$

bright optical soliton solution

$$q(x,t) = \frac{\tilde{B}}{\cosh^{\frac{2}{n}}\left[B(x - ct)\right]}, \qquad (31)$$

singular soliton solution

$$q(x,t) = \frac{\tilde{C}}{\sinh^{\frac{2}{n}}\left[C(x - ct)\right]}, \qquad (32)$$

where

$$\tilde{A} = 2A\sqrt{\tau_1}, \quad \tilde{B} = \left(\tau_1(\alpha_2 - \alpha_1)\right)^{\frac{1}{n}}, \quad B = \frac{1}{2}\frac{\sqrt{\alpha_2 - \alpha_1}}{A},$$

$$\tilde{C} = \left(\tau_1(\alpha_1 - \alpha_2)\right)^{\frac{1}{n}}, \quad C = \frac{1}{2}\frac{\sqrt{\alpha_1 - \alpha_2}}{A}.$$

Here, $\tilde{B}$ and $\tilde{C}$ are the amplitudes of the solitons, while c is the velocity; B and C is the inverse width of the solitons. Thus, we can say that the solitons exist for $\tau_1 > 0$.

On the other hand, if we take $\tau_0 = -\tau_1 \alpha_3$ and $\eta_0 = 0$, the Jacobi elliptic function solution (29) can be written in the form

$$q_i(x,t) = \tilde{D}\, sn^{\frac{2}{n}}\left[D_i(x-ct), \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}\right], \quad (33)$$

where $\tilde{D} = \left(\tau_1(\alpha_2 - \alpha_3)\right)^{\frac{1}{n}}$ and

$$D_i = \frac{(-1)^i \sqrt{\alpha_1 - \alpha_3}}{2A}, (i = 1,2).$$

Remark 1. The solutions (30)-(33) obtained by using the extended trial equation method for Eq. (1) have been checked by Mathematica. To our knowledge, the rational function solution, the singular optical soliton solution and the Jacobi elliptic function solutions, which we find in this paper, are not shown in the previous literature. These results are new exact solutions of Eq. (1).

Remark 2. When the modulus $l \to 1$, then the solution (33) can be converted into dark optical soliton solutions of the generalized regularized long wave equation

$$q_i(x,t) = \tilde{D}\, \tanh^{\frac{2}{n}}\left[D_i(x-ct)\right], \quad (34)$$

where $\alpha_1 = \alpha_2$, and c represents the velocity of the dark soliton.

Case 2. If we take $\varepsilon = 0, \delta = 2$ and $\theta = 4$ then

$$(v')^2 = \frac{\left(\xi_0 + \Gamma\xi_1 + \Gamma^2\xi_2 + \Gamma^3\xi_3 + \Gamma^4\xi_4\right)\left(\tau_1 + 2\Gamma\tau_2\right)^2}{\zeta_0},$$

$$v'' = \frac{4\tau_2\left(\xi_0 + \Gamma\xi_1 + \Gamma^2\xi_2 + \Gamma^3\xi_3 + \Gamma^4\xi_4\right) + \left(\xi_1 + 2\Gamma\xi_2 + 3\Gamma^2\xi_3 + 4\Gamma^3\xi_4\right)}{2\zeta_0\left(\tau_1 + 2\Gamma\tau_2\right)^{-1}}, \quad (35)$$

where $\xi_4 \neq 0, \zeta_0 \neq 0$. Respectively, solving the algebraic equation system (10) yields as follows:

$$\xi_2 = \frac{\xi_1^2}{3\xi_0}, \quad \xi_3 = \frac{\xi_1^3}{24\xi_0^2}, \quad \xi_4 = \frac{\xi_1^4}{576\xi_0^3}, \quad \zeta_0 = -\frac{b(n+2)\xi_1^3}{24an^2\xi_0^2\tau_1},$$

$$\tau_0 = \frac{2\xi_0\tau_1}{\xi_1}, \quad \tau_2 = \frac{\xi_1\tau_1}{12\xi_9}, \quad c = -\frac{2a\xi_0\tau_1}{(n+2)\xi_1}, \quad (36)$$

where ξ_0, ξ_1 and τ_1 are free parameters. Substituting these results into (6) and (11), we have

$$\pm(\eta - \eta_0) = 2A_1 \int \frac{d\Gamma}{\sqrt{\Gamma^4 + \dfrac{\xi_3}{\xi_4}\Gamma^3 + \dfrac{\xi_2}{\xi_4}\Gamma^2 + \dfrac{\xi_1}{\xi_4}\Gamma + \dfrac{\xi_0}{\xi_4}}}, \quad (37)$$

where $A_1 = \sqrt{-\dfrac{6b(n+2)\xi_0}{an^2\xi_1\tau_1}}$. Integrating (37), we obtain the following exact solutions to the Eq. (1).

When $\Phi(\Gamma) = (\Gamma - \alpha_1)^4$, we have

$$\pm(\eta - \eta_0) = -\frac{2A_1}{\Gamma - \alpha_1}. \quad (38)$$

While $\Phi(\Gamma) = (\Gamma - \alpha_1)^3 (\Gamma - \alpha_2)$, and $\alpha_1 > \alpha_2$, we get

$$\pm(\eta - \eta_0) = \frac{4A_1}{\alpha_1 - \alpha_2} \sqrt{\frac{\Gamma - \alpha_2}{\Gamma - \alpha_1}}. \tag{39}$$

When $\Phi(\Gamma) = (\Gamma - \alpha_1)^2 (\Gamma - \alpha_2)^2$, and $\alpha_2 > \alpha_1$, we obtain

$$\pm(\eta - \eta_0) = \frac{2A_1}{\alpha_1 - \alpha_2} \ln\left|\frac{\Gamma - \alpha_1}{\Gamma - \alpha_2}\right|. \tag{40}$$

While $\Phi(\Gamma) = (\Gamma - \alpha_1)^2 (\Gamma - \alpha_2)(\Gamma - \alpha_3)$, and $\alpha_1 > \alpha_2 > \alpha_3$, we obtain

$$\pm(\eta - \eta_0) = \frac{4A_1}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}} \ln\left|\frac{\sqrt{(\Gamma - \alpha_2)(\alpha_1 - \alpha_3)} - \sqrt{(\Gamma - \alpha_3)(\alpha_1 - \alpha_2)}}{\sqrt{(\Gamma - \alpha_2)(\alpha_1 - \alpha_3)} + \sqrt{(\Gamma - \alpha_3)(\alpha_1 - \alpha_2)}}\right|. \tag{41}$$

When $\Phi(\Gamma) = (\Gamma - \alpha_1)(\Gamma - \alpha_2)(\Gamma - \alpha_3)(\Gamma - \alpha_4)$, and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$, we find

$$\pm(\eta - \eta_0) = \frac{4A_1}{\sqrt{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)}} F(\varphi_1, l_1), \tag{42}$$

where

$$\varphi_1 = A\text{rcsin}\sqrt{\frac{(\Gamma - \alpha_2)(\alpha_1 - \alpha_4)}{(\Gamma - \alpha_1)(\alpha_2 - \alpha_4)}},$$

$$l_1^2 = \frac{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}. \tag{43}$$

Also $\alpha_1, \alpha_2, \alpha_3$ and α_4 are the roots of the polynomial equation

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4} = 0. \tag{44}$$

Substituting the solutions (38)–(42) into (5) and (14), we have

$$q(x,t) = \left[\tau_0 + \tau_1\alpha_1 \pm \frac{2\tau_1 A_1}{x + \frac{2a\xi_0\tau_1}{(n+2)\xi_1}t - \eta_0} + \tau_2\left(\alpha_1 \pm \frac{2A_1}{x + \frac{2a\xi_0\tau_1}{(n+2)\xi_1}t - \eta_0}\right)^2 \right]^{\frac{1}{n}}, \tag{45}$$

$$q(x,t) = \left[\begin{array}{l} \tau_0 + \tau_1\alpha_1 + \dfrac{16A_1^2(\alpha_2-\alpha_1)\tau_1}{16A_1^2 - \left[(\alpha_1-\alpha_2)\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right)\right]^2} \\[4em] +\tau_2\left(\alpha_1 + \dfrac{16A_1^2(\alpha_2-\alpha_1)}{16A_1^2 - \left[(\alpha_1-\alpha_2)\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right)\right]^2}\right)^2 \end{array} \right]^{\frac{1}{n}}, \tag{46}$$

$$q(x,t) = \left[\begin{array}{l} \tau_0 + \tau_1\alpha_2 + \dfrac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\dfrac{\alpha_1-\alpha_2}{2A_1}\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right)\right]-1} \\[4em] +\tau_2\left(\alpha_2 + \dfrac{(\alpha_2-\alpha_1)}{\exp\left[\dfrac{\alpha_1-\alpha_2}{2A_1}\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right)\right]-1}\right)^2 \end{array} \right]^{\frac{1}{n}}, \tag{47}$$

$$q(x,t) = \left[\begin{array}{l} \tau_0 + \tau_1\alpha_1 + \dfrac{(\alpha_1-\alpha_2)\tau_1}{\exp\left[\dfrac{\alpha_1-\alpha_2}{2A_1}\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right)\right]-1} \\[4em] +\tau_2\left(\alpha_1 + \dfrac{(\alpha_1-\alpha_2)}{\exp\left[\dfrac{\alpha_1-\alpha_2}{2A_1}\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right)\right]-1}\right)^2 \end{array} \right]^{\frac{1}{n}}, \tag{48}$$

$$q(x,t) = \left[\begin{array}{l} \tau_0 + \tau_1\alpha_1 - \dfrac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\dfrac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2A_1}\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right)\right]} \\[4em] +\tau_2\left(\alpha_1 - \dfrac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\dfrac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2A_1}\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right)\right]}\right)^2 \end{array} \right]^{\frac{1}{n}}, \tag{49}$$

$$q(x,t) = \left[\begin{array}{l} \tau_0 + \tau_1\alpha_2 + \dfrac{\tau_1(\alpha_1-\alpha_2)}{\dfrac{\alpha_1-\alpha_4}{\alpha_4-\alpha_2}\,sn^2\left[\dfrac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{4A_1}\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right),\dfrac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]+1} \\[3em] +\tau_2\left(\alpha_2+\dfrac{(\alpha_1-\alpha_2)}{\dfrac{\alpha_1-\alpha_4}{\alpha_4-\alpha_2}\,sn^2\left[\dfrac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{4A_1}\left(x+\dfrac{2a\xi_0\tau_1}{(n+2)\xi_1}t-\eta_0\right),\dfrac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]+1}\right)^2 \end{array} \right]^{\frac{1}{n}}. \tag{50}$$

For simplicity, if we take $\eta_0 = 0$, then we can write the solutions (45)-(50) as follows:

$$q(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 \pm \frac{2A_1}{x-ct}\right)^i\right]^{\frac{1}{n}}, \tag{51}$$

$$q(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 \pm \frac{16A_1^2(\alpha_1-\alpha_2)}{16A_1^2-\left[(\alpha_1-\alpha_2)(x-ct)\right]^2}\right)^i\right]^{\frac{1}{n}}, \tag{52}$$

$$q(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_2 + \frac{(\alpha_2-\alpha_1)}{\exp\left[B_1(x-ct)\right]-1}\right)^i\right]^{\frac{1}{n}}, \tag{53}$$

$$q(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 + \frac{(\alpha_1-\alpha_2)}{\exp\left[B_1(x-ct)\right]-1}\right)^i\right]^{\frac{1}{n}}, \tag{54}$$

$$q(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 - \frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[B_2(x-ct)\right]}\right)^i\right]^{\frac{1}{n}}, \tag{55}$$

$$q(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_2 + \frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\,sn^2(\tilde{\varphi},l_1)}\right)^i\right]^{\frac{1}{n}}, \tag{56}$$

where $\quad B_1 = \dfrac{\alpha_1-\alpha_2}{2A_1} \quad , \quad B_2 = \dfrac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2A_1} \quad , \quad \tilde{\varphi} = \dfrac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{4A_1}(x-ct) \quad ,$

$l_1^2 = \dfrac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}$. Here, c is the velocity, and B_1 and B_2 are the inverse widths of the solitons.

Remark 3. The solutions (51)-(56) obtained by using the extended trial equation method for Eq. (1) have not been found in the literature, and these results are new.

Case 3. If we take $\varepsilon = 0$, $\delta = 3$ and $\theta = 5$ then

$$(v')^2 = \frac{\left(\xi_0 + \Gamma\xi_1 + \Gamma^2\xi_2 + \Gamma^3\xi_3 + \Gamma^4\xi_4 + \Gamma^5\xi_5\right)}{\zeta_0\left(\tau_1 + 2\tau_2\Gamma + 3\tau_3\Gamma^2\right)^{-2}}, \tag{57}$$

where $\xi_5 \neq 0$, $\zeta_0 \neq 0$. Respectively, solving the algebraic equation system (10) yields

$$\xi_0 = \frac{6an^2\zeta_0\tau_2^5 + 729b(n+2)\xi_1\tau_2\tau_3^3}{4374b(n+2)\tau_3^4}, \qquad \xi_2 = \frac{243b(n+2)\xi_2\tau_3^3 - 10an^2\zeta_0\tau_2^4}{162b(n+2)\tau_2\tau_3^2},$$

$$\xi_3 = -\frac{20an^2\zeta_0\tau_2^2}{81b(n+2)\tau_3}, \qquad \xi_4 = -\frac{10an^2\zeta_0\tau_2}{27b(n+2)}, \quad \xi_5 = -\frac{2an^2\zeta_0\tau_3}{9b(n+2)}, \quad \tau_0 = \frac{\tau_2^3}{27\tau_3^2}, \quad \tau_1 = \frac{\tau_2^2}{3\tau_3},$$

$$c = \frac{10an^2\tau_0\tau_2^4 + 729b(n+2)\xi_1\tau_3^3}{54n^2(n+2)\tau_0\tau_2\tau_3^2}. \tag{58}$$

where ξ_1, ζ_0, τ_2 and τ_3 are free parameters. Substituting these results into Eqs. (6) and (11), we get

$$\pm(\eta - \eta_0) = 3A_2 \int \frac{d\Gamma}{\sqrt{\Gamma^5 + \dfrac{\xi_4}{\xi_5}\Gamma^4 + \dfrac{\xi_3}{\xi_5}\Gamma^3 + \dfrac{\xi_2}{\xi_5}\Gamma^2 + \dfrac{\xi_1}{\xi_5}\Gamma + \dfrac{\xi_0}{\xi_5}}}, \tag{59}$$

where $A_2 = \sqrt{-\dfrac{b(n+2)}{2an^2\tau_3}}$. Integrating (59), we obtain the exact solutions to the Eq. (1) as follows:

$$\pm(\eta - \eta_0) = -\frac{2A_2}{\sqrt{(\Gamma - \alpha_1)^3}}, \tag{60}$$

$$\pm(\eta - \eta_0) = \frac{3A_2 Arc\tanh\left[\sqrt{\dfrac{\Gamma - \alpha_2}{\alpha_1 - \alpha_2}}\right]}{(\alpha_1 - \alpha_2)^{\frac{3}{2}}} - \frac{3A_2\sqrt{\Gamma - \alpha_2}}{(\alpha_1 - \alpha_2)(\Gamma - \alpha_1)}, \tag{61}$$

$$\pm(\eta - \eta_0) = -\frac{6A_2 Arc\tan\left[\sqrt{\dfrac{\Gamma - \alpha_1}{\alpha_1 - \alpha_2}}\right]}{(\alpha_1 - \alpha_2)^{\frac{3}{2}}} - \frac{6A_2}{\sqrt{\Gamma - \alpha_1}(\alpha_1 - \alpha_2)}, \tag{62}$$

$$\pm(\eta - \eta_0) = \frac{6A_2 Arc\tanh\left[\sqrt{\dfrac{\Gamma - \alpha_3}{\alpha_2 - \alpha_3}}\right]}{\alpha_1 - \alpha_2}\left(\frac{1}{\sqrt{\alpha_2 - \alpha_3}} - \frac{1}{\sqrt{\alpha_1 - \alpha_3}}\right), \tag{63}$$

$$\pm(\eta - \eta_0) = \frac{-6A_2\left(\sqrt{(\Gamma - \alpha_2)(\Gamma - \alpha_3)} + i\left(E(\varphi_2, l_2) - F(\varphi_2, l_2)\right)\right)}{\sqrt{\Gamma - \alpha_1}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}, \tag{64}$$

$$\pm(\eta - \eta_0) = \frac{-6iA_2\left(F(\varphi_2, l_2) - \Pi(\varphi_2, n, l_2)\right)}{\sqrt{\alpha_2 - \alpha_3}(\alpha_1 - \alpha_2)}, \tag{65}$$

where

$$E(\varphi_2, l_2) = \int_0^{\varphi_1} \sqrt{1 - l_1^2 \sin^2 \Psi} \, d\Psi, \quad F(\varphi_i, l_i) = \int_0^{\varphi_i} \frac{d\Psi}{\sqrt{1 - l_i^2 \sin^2 \Psi}}, \quad i = 2, 3.$$

$$\Pi(\varphi_3, n, l_3) = \int_0^{\varphi_2} \frac{d\Psi}{\left(1 + n \sin^2 \Psi\right)\sqrt{1 - l_2^2 \sin^2 \Psi}},$$

$$\varphi_2 = -Arc\sin\sqrt{\frac{\Gamma - \alpha_1}{\alpha_2 - \alpha_1}}, \qquad l_2^2 = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3},$$

$$\varphi_3 = -Arc\sin\sqrt{\frac{\alpha_3 - \alpha_2}{\Gamma - \alpha_2}}, \qquad l_3^2 = \frac{\alpha_2 - \alpha_4}{\alpha_2 - \alpha_3}, \qquad n = \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_3}. \tag{66}$$

Also α_1, α_2, α_3, α_4 and α_5 are the roots of the polynomial equation

$$\Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma + \frac{\xi_0}{\xi_5} = 0. \tag{67}$$

Case 4. If we take $\varepsilon = 1$, $\delta = 1$ and $\theta = 4$ then

$$\left(v'\right)^2 = \frac{\left(\xi_0 + \Gamma\xi_1 + \Gamma^2\xi_2 + \Gamma^3\xi_3 + \Gamma^4\xi_4\right)\tau_1^2}{\zeta_0 + \Gamma\zeta_1},$$

$$v'' = \frac{\left(\left(\zeta_0 + \Gamma\zeta_1\right)\left(\xi_1 + 2\Gamma\xi_2 + 3\Gamma^2\xi_3 + 4\Gamma^3\xi_4\right) - \zeta_1\left(\xi_0 + \Gamma\xi_1 + \Gamma^2\xi_2 + \Gamma^3\xi_3 + \Gamma^4\xi_4\right)\right)\tau_1}{2\left(\zeta_0 + \Gamma\zeta_1\right)^2}, \tag{68}$$

where $\xi_4 \neq 0$, $\zeta_1 \neq 0$. Respectively, solving the algebraic equation system (10) yields

$$\xi_1 = -\frac{4a^2n^4\zeta_0^2\tau_0^3 + b(n+2)\xi_0\left(b(n+2)\xi_4\tau_0 - 4an^2\zeta_0\tau_1^2\right)}{2abn^2(n+2)\zeta_0\tau_0\tau_1},$$

$$\zeta_1 = -\frac{b(n+2)\xi_4}{2an^2\tau_1}, \quad \xi_2 = \frac{abn^2(n+2)\zeta_0\left(\xi_4\tau_0^4 + \xi_0\tau_1^4\right) - b^2(n+2)^2\xi_0\xi_4\tau_0\tau_1^2 - 4a^2n^4\zeta_0^2\tau_0^3\tau_1^2}{abn^2(n+2)\zeta_0\tau_0^2\tau_1^2},$$

$$\xi_3 = \frac{4abn^2(n+2)\zeta_0\xi_4\tau_0^3 - \tau_1^2\left(b^2(n+2)^2\xi_0\xi_4 + 4a^2n^4\zeta_0^2\tau_0^2\right)}{2abn^2(n+2)\zeta_0\tau_0^2\tau_1}, \quad c = \frac{2an^2\zeta_0\tau_0^3 + b(n+2)\xi_0\tau_1^2}{n^2(n+2)\zeta_0\tau_0^2}. \tag{69}$$

where ξ_0, ξ_4, τ_0, ζ_0 and τ_1 are free parameters. Substituting these results into Eqs. (6) and (11), we get

$$\pm(\eta-\eta_0) = A \int \sqrt{\dfrac{\Gamma + \dfrac{\zeta_0}{\zeta_1}}{\Gamma^4 + \dfrac{\xi_3}{\xi_4}\Gamma^3 + \dfrac{\xi_2}{\xi_4}\Gamma^2 + \dfrac{\xi_1}{\xi_4}\Gamma + \dfrac{\xi_0}{\xi_4}}}\, d\Gamma.$$

(70)

Integrating (70), we obtain the exact solutions to the Eq. (1) as follow:

$$\pm(\eta-\eta_0) = -A\sqrt{\dfrac{\zeta_1}{\zeta_0+\zeta_1\alpha_1}} Arc\tanh\left[\sqrt{\dfrac{\zeta_0+\zeta_1\Gamma}{\zeta_0+\zeta_1\alpha_1}}\right]$$
$$-\dfrac{A}{\Gamma-\alpha_1}\sqrt{\dfrac{\zeta_0+\zeta_1\Gamma}{\zeta_1}},$$

(71)

$$\pm(\eta-\eta_0) = \dfrac{2A}{\alpha_1-\alpha_2}\left(\begin{array}{l}\sqrt{\dfrac{\zeta_0+\zeta_1\alpha_1}{\zeta_1}} Arc\tanh\left[\sqrt{\dfrac{\zeta_0+\zeta_1\Gamma}{\zeta_0+\zeta_1\alpha_1}}\right]\\ +\sqrt{\zeta_0+\zeta_1\alpha_2} Arc\tanh\left[\sqrt{\dfrac{\zeta_0+\zeta_1\Gamma}{\zeta_0+\zeta_1\alpha_2}}\right]\end{array}\right),$$

(72)

$$\pm(\eta-\eta_0) = 2A\left(\begin{array}{l}\sqrt{\dfrac{(\Gamma-\alpha_1)\zeta_0+\zeta_1\Gamma}{\zeta_1(\Gamma-\alpha_2)^2}}\\ +i\sqrt{\alpha_1-\alpha_2}\left(E(\varphi_4,l_4)-F(\varphi_4,l_4)\right)\end{array}\right),$$

(73)

$$\pm(\eta-\eta_0) = \dfrac{-2A}{\sqrt{\alpha_2-\alpha_1}} E(\varphi_5,l_5),$$

(74)

$$\pm(\eta-\eta_0) = \dfrac{-2iA(\zeta_0+\zeta_1\alpha_1)}{(\alpha_1-\alpha_2)\sqrt{\zeta_1(\zeta_0+\zeta_1\alpha_1)}}\left(E(\varphi_6,l_6)-\Pi(\varphi_6,n,l_6)\right),$$

(75)

where

$$\varphi_4 = iArc\sinh\sqrt{\dfrac{\zeta_1(\Gamma-\alpha_1)}{\zeta_0+\zeta_1\alpha_1}},$$

$$l_4^2 = \dfrac{\zeta_0+\zeta_1\alpha_1}{\zeta_1(\alpha_1-\alpha_2)},$$

(76)

$$\varphi_5 = Arc\sinh\left[\sqrt{\dfrac{\alpha_1-\alpha_2}{\Gamma-\alpha_1}}\right],$$

$$l_5^2 = \dfrac{\zeta_0+\zeta_1\alpha_1}{\zeta_1(\alpha_1-\alpha_2)},$$

(77)

$$\varphi_6 = iArc\sinh\sqrt{\dfrac{\zeta_0+\zeta_1\alpha_2}{\zeta_1(\Gamma-\alpha_2)}},$$

$$l_6^2 = \dfrac{\zeta_1(\alpha_2-\alpha_3)}{\zeta_0+\zeta_1\alpha_2}, \qquad n = \dfrac{\zeta_1(\alpha_2-\alpha_1)}{\zeta_0+\zeta_1\alpha_2}.$$

(78)

Case 5. If we take $\varepsilon=1$, $\delta=2$ and $\theta=5$, then

$$(v')^2 = \dfrac{\left(\xi_0+\Gamma\xi_1+\Gamma^2\xi_2+\Gamma^3\xi_3+\Gamma^4\xi_4+\Gamma^5\xi_5\right)}{\left(\zeta_0+\Gamma\zeta_1\right)\left(\tau_1+2\Gamma\tau_2\right)^{-2}},$$

(79)

where $\xi_5 \neq 0$, $\zeta_1 \neq 0$. Respectively, solving the algebraic equation system (10) yields

$$\xi_0 = -\dfrac{a\zeta_0\left(an^3\zeta_0\tau_1^2+2bn(n+2)\xi_2\tau_2\right)^2}{2b(n+2)\tau_2\left(an^2\zeta_0\tau_2-2b(n+2)\xi_4\right)}, \quad \xi_5 = \dfrac{\tau_2\left(an^2\zeta_0\tau_2+2b(n+2)\xi_4\right)}{4b(n+2)\tau_1},$$

$$\xi_1 = \frac{4b^2(n+2)^2\xi_4^2\tau_1^2 + 4b(n+2)\xi_4\tau_2\left(2b(n+2)\xi_2\tau_2 - an^2\zeta_0\tau_1^2\right) + an^2\zeta_0\tau_2^2\left(5an^2\zeta_0\tau_1^2 + 4b(n+2)\xi_2\tau_2\right)}{4b(n+2)\tau_1\tau_2\left(2b(n+2)\xi_4 - an^2\zeta_0\tau_2\right)},$$

$$\xi_3 = \frac{\left(2b(n+2)\xi_2\tau_2 + an^2\zeta_0\tau_1^2\right)\left(4b^2(n+2)^2\xi_2\xi_4\tau_2 + 5a^2n^4\zeta_0^2\tau_1^2\tau_2 - 2abn^2(n+2)\zeta_0\left(3\xi_4\tau_1^2 - \xi_2\tau_2^2\right)\right)}{4b(n+2)\tau_1\tau_2\left(an^2\zeta_0\tau_2 - 2b(n+2)\xi_4\right)^2},$$

$$\tau_0 = \frac{an^2\zeta_0\tau_1^2 + 2b(n+2)\xi_2\tau_2}{2b(n+2)\xi_4 - an^2\zeta_0\tau_2},$$

$$\zeta_1 = -\frac{2b(n+2)\xi_4 + an^2\zeta_0\tau_2}{2an^2\tau_1}, \; c = -\frac{a\left(\tau_2\left(5an^2\zeta_0\tau_1^2 + 8b(n+2)\xi_2\tau_2\right) - 2b(n+2)\xi_4\tau_1^2\right)}{2(n+2)\tau_2\left(an^2\zeta_0\tau_2 - 2b(n+2)\xi_4\right)}, \tag{80}$$

where ξ_2, ξ_4, τ_1, ζ_0 and τ_2 are free parameters. Substituting these results into (6) and (11), we have

$$\pm(\eta - \eta_0) = A_3 \int \sqrt{\frac{\Gamma + \dfrac{\zeta_0}{\zeta_1}}{\Gamma^5 + \dfrac{\xi_4}{\xi_5}\Gamma^4 + \dfrac{\xi_3}{\xi_5}\Gamma^3 + \dfrac{\xi_2}{\xi_5}\Gamma^2 + \dfrac{\xi_1}{\xi_5}\Gamma + \dfrac{\xi_0}{\xi_5}}} \, d\Gamma, \tag{81}$$

where $A_3 = \sqrt{-\dfrac{b(n+2)}{an^2\tau_2}}$. Integrating (81), we obtain the exact solutions to the Eq. (1) as follow:

When $\Phi(\Gamma) = (\Gamma - \alpha_1)^5$, we have

$$\pm(\eta - \eta_0) = -\frac{2A_3}{3\sqrt{\zeta_1}(\zeta_0 + \zeta_1\alpha_1)^2}\left(\frac{\zeta_0 + \Gamma\zeta_1}{\Gamma - \alpha_1}\right)^{\frac{3}{2}}. \tag{82}$$

If we take $\Phi(\Gamma) = (\Gamma - \alpha_1)^4(\Gamma - \alpha_2)$ and $\alpha_1 > \alpha_2$, then we get

$$\pm(\eta - \eta_0) = \frac{-A_3}{\alpha_1 - \alpha_2}\left[\begin{array}{l}\dfrac{(\zeta_0 + \zeta_1\alpha_2)}{2\sqrt{\zeta_1(\alpha_1 - \alpha_2)(\zeta_0 + \zeta_1\alpha_1)}}\ln\left|\dfrac{\Gamma - \alpha_1}{K(\Gamma) + \zeta_0(\alpha_1 - 2\alpha_2) - \zeta_1\alpha_2\alpha_1 + L(\Gamma)}\right| \\[4mm] +\dfrac{1}{\Gamma - \alpha_1}\sqrt{\dfrac{(\zeta_0 + \zeta_1\Gamma)(\Gamma - \alpha_2)}{\zeta_1}}\end{array}\right], \tag{83}$$

where

$$K(\Gamma) = \left(\zeta_0 + 2\zeta_1\alpha_1 - \zeta_1\alpha_2\right)\Gamma,$$

$$L(\Gamma) = 2\sqrt{(\zeta_0 + \zeta_1\Gamma)(\zeta_0 + \zeta_1\alpha_1)(\Gamma - \alpha_2)(\alpha_1 - \alpha_2)}.$$

While $\Phi(\Gamma) = (\Gamma - \alpha_1)^3(\Gamma - \alpha_2)^2$ and $\alpha_1 > \alpha_2$, we obtain

$$\pm(\eta - \eta_0) = \frac{-2A_3}{\alpha_1 - \alpha_2}\left[\sqrt{\frac{\zeta_0 + \zeta_1\Gamma}{\zeta_1(\Gamma - \alpha_1)}} + \sqrt{\frac{\zeta_0 + \zeta_1\alpha_2}{\zeta_1(\alpha_1 - \alpha_2)}}Arc\tan\left(\sqrt{\frac{(\Gamma - \alpha_1)(\zeta_0 + \zeta_1\alpha_2)}{(\alpha_1 - \alpha_2)(\zeta_0 + \zeta_1\Gamma)}}\right)\right]. \tag{84}$$

If we take $\Phi(\Gamma) = (\Gamma - \alpha_1)^2(\Gamma - \alpha_2)^2(\Gamma - \alpha_3)$ and $\alpha_1 > \alpha_2 > \alpha_3$, then we get

$$\pm(\eta - \eta_0) = \frac{-A_3}{(\alpha_1 - \alpha_3)\sqrt{\zeta_1}}\left[Y\ln\left|\frac{\alpha_2 - \Gamma}{P(\Gamma) + \zeta_0(\alpha_2 - 2\alpha_3) - \zeta_1\alpha_2\alpha_3 + Q(\Gamma)}\right| + Z\ln\left|\frac{R(\Gamma) + \zeta_0(\alpha_1 - 2\alpha_3) - \zeta_1\alpha_1\alpha_3 + S(\Gamma)}{\Gamma - \alpha_2}\right|\right], \tag{85}$$

where

$$Y = \sqrt{\frac{\zeta_0 + \zeta_1\alpha_2}{\alpha_2 - \alpha_3}},$$

$$P(\Gamma) = 2\sqrt{(\zeta_0 + \zeta_1\Gamma)(\zeta_0 + \zeta_1\alpha_2)(\Gamma - \alpha_3)(\alpha_2 - \alpha_3)}, \quad Q(\Gamma) = (\zeta_0 + 2\zeta_1\alpha_2 - \zeta_1\alpha_3)\Gamma, \tag{86}$$

$$Z = \sqrt{\frac{\zeta_0 + \zeta_1\alpha_1}{\alpha_1 - \alpha_3}},$$

$$R(\Gamma) = 2\sqrt{(\zeta_0 + \zeta_1\Gamma)(\zeta_0 + \zeta_1\alpha_1)(\Gamma - \alpha_3)(\alpha_1 - \alpha_3)},$$

$$S(\Gamma) = (\zeta_0 + 2\zeta_1\alpha_1 - \zeta_1\alpha_3)\Gamma. \tag{87}$$

When $\Phi(\Gamma) = (\Gamma - \alpha_1)^3(\Gamma - \alpha_2)^2(\Gamma - \alpha_3)$ and $\alpha_1 > \alpha_2 > \alpha_3$, then we obtain

$$\pm(\eta - \eta_0) = \frac{-2A_3}{\alpha_1 - \alpha_3}\sqrt{\frac{\zeta_0 + \zeta_1\alpha_3}{\zeta_1(\alpha_1 - \alpha_2)}}E(\varphi_7, l_7), \tag{88}$$

$$\varphi_7 = \arcsin\sqrt{\frac{(\Gamma - \alpha_3)(\alpha_2 - \alpha_1)}{(\Gamma - \alpha_1)(\alpha_2 - \alpha_3)}},$$

$$l_7^2 = \frac{(\zeta_0 + \zeta_1\alpha_1)(\alpha_3 - \alpha_2)}{(\alpha_1 - \alpha_2)(\zeta_0 + \zeta_1\alpha_3)}. \tag{89}$$

If we take $\Phi(\Gamma) = (\Gamma - \alpha_1)^2(\Gamma - \alpha_2)(\Gamma - \alpha_3)(\Gamma - \alpha_4)$ and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$, then we get

$$\pm(\eta - \eta_0) = \frac{2A_3(\alpha_2 - \alpha_4)\left(\dfrac{\zeta_0 + \zeta_1\Gamma}{\alpha_1 - \alpha_4}\Pi(\varphi_8, n, l_8) - \dfrac{\zeta_0 + \zeta_1\alpha_2}{\alpha_2 - \alpha_4}F(\varphi_8, l_8)\right)}{(\alpha_1 - \alpha_2)\sqrt{\zeta_1(\alpha_2 - \alpha_3)(\zeta_0 + \zeta_1\alpha_4)}},$$

(90)

where

$$\varphi_8 = \arcsin\sqrt{\frac{(\Gamma - \alpha_4)(\alpha_3 - \alpha_2)}{(\Gamma - \alpha_2)(\alpha_3 - \alpha_4)}}, \quad l_8^2 = \frac{(\zeta_0 + \zeta_1\alpha_2)(\alpha_4 - \alpha_3)}{(\alpha_2 - \alpha_3)(\zeta_0 + \zeta_1\alpha_4)},$$

$$n = -\frac{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)}{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}.$$

(91)

3. Conclusion

In this study, we applied extended trial equation method to the KdV equation with power-law nonlinearity and constructed the bright and dark optical soliton solutions, elliptic function and Jacobi elliptic function solutions to this nonlinear physical problem. These solutions will be very useful for the study of nonlinear optics, nonlinear optical materials, fluid dynamics, plasma physics and many other areas. The results obtained with extended trial equation method are new and explicit forms. Afterwards, the focus will be on the application of supplemental integration techniques to acquire dark and singular optical solitons together with bright-dark combo optical solitons. Consequently, this method can be used as a suitable technique for nonlinear partial differential equation to obtain different types of optical solitons.

References

[1] M. L. Wang, X. Z. Li, Phys. Lett. A., **343**, 48 (2005).
[2] W. W. Li, Y. Tian, Z. Zhang, Appl. Math. Comput., **219**, 1135 (2012).
[3] N. K. Vitanov, Z. I. Dimitrova, Commun. Nonlinear Sci. Numer. Simulat. **15**(10), 2836 (2010).
[4] N. K. Vitanov, Z. I. Dimitrova, H. Kantz, Appl. Math. Comput. **216**(9) 2587 (2010).
[5] C. M. Khalique, A. Biswas, J. Electromagn. Waves Appl. **23,** 963 (2009).
[6] J. Zhang, Int. J. Theor. Phys. **37**, 1541 (1998).
[7] D. C. Lu, C. L. Liu, Comput. Math. Appl. **217**, 1404 (2010).
[8] A. H. Bhrawy, M. A. Abdelkawy, Cent. Eur. J. Phys. (2013).
[9] M. Hayek, Appl. Math. Comput. **217,** 212 (2010).
[10] X. Liu, W. Zhang, Z. Li, Adv. Appl. Math. Mech. **4**, 122 (2010).
[11] C. S. Liu, Commun. Theor. Phys. **45**, 219 (2010).
[12] C. S. Liu, Acta. Phys. Sin. **54**, 2505 (2005).
[13] C. Y. Jun, Comput. Math. Appl. **62**, 3987 (2012).
[14] H. W. Schurmann, V. S. Serov, J. Nickel, Int. J. Theor. Phys. **45**, 1093 (2006**)**.
[15] J. Nickel, H. W. Schurmann, Int. J. Theor. Phys. **45**, 1825 (2006).
[16] C. S. Liu, Comput. Phys. Commun. **181**, 317 (2010).
[17] C. Y. Jun, Math. Comput. Model. **56,** 43 (2012).
[18] Y. Pandir, Y. Gurefe, U. Kadak, E. Misirli, Abstr. Appl. Anal. **2012**, (2012).
[19] Y. Pandir, Y. Gurefe, E. Misirli, Phys. Scr. **87**, (2013).
[20] Y. Gurefe, E. Misirli, A. Sonmezoglu, M. Ekici, Appl. Math. Comput. **219**, 5253 (2013).
[21] Yusuf Pandir, Pramana- Journal of Physics **82**(6), 949 (2014).
[22] Hasan Bulut, Yusuf Pandir, Seyma Tuluce Demiray, Waves in Random and Complex Media **24**(4), 439 (2014).
[23] W. Zhang, Q. Chang, B. Jiang, Chaos Soliton Fract. **13,** 311 (2013).

[*]Corresponding author: yusufpandir@gmail.com