Optical solitons in birefringent fibers with Riccati equation method

MOHAMMAD MIRZAZADEH[a], MOHAMMAD F. MAHMOOD[b], FAYEQUA B. MAJID[c], ANJAN BISWAS[d,e,*], MILIVOJ BELIC[f]

[a]*Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, P.C. 44891-63157, Rudsar-Vajargah, Iran*
[b]*Department of Mathematics, Howard University, Washington, DC-20059, USA*
[c]*Department of Physics, Chemistry & Mathematics, Alabama A & M University, Normal, AL-35762, USA*
[d]*Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA*
[e]*Faculty of Science, Department of Mathematics, King Abdulaziz University, Jeddah-21589, Saudi Arabia*
[f]*Science Program, Texas A & M University at Qatar, PO Box 23874, Doha, Qatar*

This paper addresses optical solitons in birefringent fibers with Kerr law nonlinearity, in presence of perturbation terms and spatio-temporal dispersion. The Riccati equation expansion algorithm is applied to secure soliton solutions to the model. Dark and singular soliton solutions are obtained for both components of the model. Additional solutions revealed, with this algorithm, are plane waves and singular periodic solutions. These are obtained as a byproduct and are not applicable to birefringent fibers.

(Received January 30, 2015; accepted June 24, 2015)

Keywords: Birefringence, Integrability, Solitons

1. Introduction

Optical solitons is a cherished area of research in the field of nonlinear optics [1-20]. These solitons appear in polarization-preserving fibers as well as birefringent fibers. When pulses are polarized due to bend, twists and external stress on these fibers, they split and this leads to differential group delay and thus the two pulses undergo polarization mode dispersion [2, 4]. With such an unwanted physical situation the governing nonlinear Schrödinger's equation (NLSE) splits to vector coupled NLSE in a birefringent fiber.

The integrability of this vector coupled NLSE is a challenging task. This paper applies a mathematical technique that is known as the Riccati equation method that integrates this coupled NLSE in presence of several perturbation terms. This integration scheme leads to dark solitons and singular soliton solutions with constraint conditions that must stay valid for these solitons to exist. Moreover, as a byproduct, other forms of nonlinear wave solutions emerge. These are plane waves and singular periodic solutions. The details are discussed in the section below. It must, however, be noted that ansatz approach was applied earlier where bright, dark and singular soliton solutions were obtained [2, 4].

2. Governing equation

The dimensionless form of the coupled NLSE with STD and Kerr law nonlinearity, with perturbation terms, is given by [2]

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + (c_1|q|^2 + d_1|r|^2)q + i\left\{\begin{array}{l}\alpha_1 q_x + \lambda_1(|q|^2 q)_x + \nu_1(|q|^2)_x q \\ +\theta_1|q|^2 q_x + \gamma_1 q_{xxx}\end{array}\right\} = 0 \tag{1}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + (c_2|r|^2 + d_2|q|^2)r + i\left\{\begin{array}{l}\alpha_2 r_x + \lambda_2(|r|^2 r)_x + \nu_2(|r|^2)_x r \\ +\theta_2|r|^2 r_x + \gamma_2 r_{xxx}\end{array}\right\} = 0 \tag{2}$$

For equations (1) and (2), $q(x,t)$ and $r(x,t)$ are wave vector fields in birefringent fibers that are complex-valued functions. for $l = 1, 2$, a_l represents the coefficients of group velocity dispersion (GVD), while b_l are the coefficients of spatio-temporal dispersion (STD). The coefficients of self-phase modulation (SPM) and cross-phase modulation (XPM) are c_l and d_l respectively. From the perturbation terms, α_l represents inter-modal dispersion; λ_l are the coefficients of self-steepening terms, ν_l and θ_l are the coefficients of nonlinear dispersion and finally γ_l gives the third order dispersion.

The consideration of STD in addition to GVD is needed in order to make the governing equation well-posed [6]. The self-steepening terms are included to avoid the formation of shock waves. Finally, the 3OD terms creep in, if GVD is low. The SPM term is due to Kerr law nonlinearity. In order to study (1) and (2), the following transformation is the starting point [2]:

$$q(x,t)=U_1(\xi)e^{i\Phi_1(x,t)} \tag{3}$$

$$r(x,t)=U_2(\xi)e^{i\Phi_2(x,t)} \tag{4}$$

where $U_l(\xi)$ represent the shape of the pulse and

$$\xi=B(x-vt) \tag{5}$$

$$\Phi_l(x,t)=-\kappa_l x+\omega_l t+\sigma_l\,,\ l=1,2 \tag{6}$$

In Eqs. (3) and (4), the functions $\Phi_l(x,t)$ represent phase components of the soliton. From the phase, κ_l are the soliton frequency ω_l are the wave numbers and σ_l are the phase constants. Finally in Eq. (5), v is the velocity of the soliton. Substituting Eqs. (3) and (4) into Eqs. (1) and (2) and decomposing into real and imaginary parts lead to

$$\begin{aligned}&\mathrm{Re1:}\\ &B^2(a_1-b_1v+3\kappa_1\gamma_1)U_1''\\ &+\begin{pmatrix}-\omega_1-a_1\kappa_1^2+b_1\kappa_1\omega_1\\ +\alpha_1\kappa_1-\gamma_1\kappa_1^3\end{pmatrix}U_1\\ &+(\lambda_1\kappa_1+\theta_1\kappa_1+c_1)U_1^3+d_1U_2^2U_1=0\end{aligned} \tag{7}$$

$$\begin{aligned}&\mathrm{Im1:}\\ &\begin{Bmatrix}(b_1\kappa_1-1)v-2a_1\kappa_1\\ +b_1\omega_1+\alpha_1-3\gamma_1\kappa_1^2\end{Bmatrix}U_1'\\ &+(3\lambda_1+2v_1+\theta_1)U_1^2U_1'+B^2\gamma_1U_1'''=0\end{aligned} \tag{8}$$

$$\begin{aligned}&\mathrm{Re2:}\\ &B^2(a_2-b_2v+3\kappa_2\gamma_2)U_2''\\ &+\begin{pmatrix}-\omega_2-a_2\kappa_2^2+b_2\kappa_2\omega_2\\ +\alpha_2\kappa_2-\gamma_2\kappa_2^3\end{pmatrix}U_2\\ &+(\lambda_2\kappa_2+\theta_2\kappa_2+c_2)U_2^3+d_2U_1^2U_2=0\end{aligned} \tag{9}$$

$$\begin{aligned}&\mathrm{Im2:}\\ &\begin{Bmatrix}(b_2\kappa_2-1)v-2a_2\kappa_2\\ +b_2\omega_2+\alpha_2-3\gamma_2\kappa_2^2\end{Bmatrix}U_2'\\ &+(3\lambda_2+2v_2+\theta_2)U_2^2U_2'+B^2\gamma_2U_2'''=0\end{aligned} \tag{10}$$

Integrating Eqs. (8) and (10) respect to ξ, then we have

$$\begin{aligned}&\begin{Bmatrix}(b_1\kappa_1-1)v-2a_1\kappa_1\\ +b_1\omega_1+\alpha_1-3\gamma_1\kappa_1^2\end{Bmatrix}U_1\\ &+\frac{1}{3}(3\lambda_1+2v_1+\theta_1)U_1^3+B^2\gamma_1U_1''=0\end{aligned} \tag{11}$$

$$\begin{aligned}&\begin{Bmatrix}(b_2\kappa_2-1)v-2a_2\kappa_2\\ +b_2\omega_2+\alpha_2-3\gamma_2\kappa_2^2\end{Bmatrix}U_2\\ &+\frac{1}{3}(3\lambda_2+2v_2+\theta_2)U_2^3+B^2\gamma_2U_2''=0\end{aligned} \tag{12}$$

By Eqs. (11) and (12), we have

$$\gamma_l=0 \tag{13}$$

$$3\lambda_l+2v_l+\theta_l=0 \tag{14}$$

$$(b_l\kappa_l-1)v-2a_l\kappa_l+b_l\omega_l+\alpha_l=0\,,\ l=1,2 \tag{15}$$

Therefore (13) shows tha this integration algorithm is applicable provided 3OD is take off from the model. Thus, the integrable model equations (1) and (2) reduces to

$$\begin{aligned}&iq_t+a_1q_{xx}+b_1q_{xt}+(c_1|q|^2+d_1|r|^2)q\\ &+i\begin{Bmatrix}\alpha_1q_x+\lambda_1(|q|^2q)_x\\ +v_1(|q|^2)_xq+\theta_1|q|^2q_x\end{Bmatrix}=0\end{aligned} \tag{16}$$

$$\begin{aligned}&ir_t+a_2r_{xx}+b_2r_{xt}+(c_2|r|^2+d_2|q|^2)r\\ &+i\begin{Bmatrix}\alpha_2r_x+\lambda_2(|r|^2r)_x\\ +v_2(|r|^2)_xr+\theta_2|r|^2r_x\end{Bmatrix}=0\end{aligned} \tag{17}$$

Next, from Eq. (15), we obtain

$$v=\frac{2a_l\kappa_l-b_l\omega_l-\alpha_l}{b_l\kappa_l-1} \tag{18}$$

Eq. (18) is the velocity of the soliton for the two components and Eq. (14) represents the constraint condition to ensure the solitons exist. Also, from (18), the following relation is imposed:

$$b_l\kappa_l\neq 1 \tag{19}$$

Then, from (18), equating the velocity of the solitons leads to the constraint condition given by

$$\begin{aligned}&(2a_1\kappa_1 - b_1\omega_1 - \alpha_1)(b_2\kappa_2 - 1) = \\ &(2a_2\kappa_2 - b_2\omega_2 - \alpha_2)(b_1\kappa_1 - 1)\end{aligned} \tag{20}$$

By Eq. (13), Eqs. (7) and (9) reduce to

$$\begin{aligned}&B^2(a_1 - b_1 v)U_1'' \\ &+ \left(-\omega_1 - a_1\kappa_1^2 + b_1\kappa_1\omega_1 + \alpha_1\kappa_1\right)U_1 \\ &+ (\lambda_1\kappa_1 + \theta_1\kappa_1 + c_1)U_1^3 + d_1U_2^2U_1 = 0\end{aligned} \tag{21}$$

$$\begin{aligned}&B^2(a_2 - b_2 v)U_2'' \\ &+ \left(-\omega_2 - a_2\kappa_2^2 + b_2\kappa_2\omega_2 + \alpha_2\kappa_2\right)U_2 \\ &+ (\lambda_2\kappa_2 + \theta_2\kappa_2 + c_2)U_2^3 + d_2U_1^2U_2 = 0\end{aligned} \tag{22}$$

3. Ricatti equation approach

In this section, the Ricatti equation approach [8] will be shown in detail to obtain the singular solutions, singular and dark soliton solutions to Eqs. (1) and (2). According to the homogeneous balance method, Eqs. (21) and (22) have the solutions in the form of

$$U_1(\xi) = A_0 + A_1\varphi(\xi) \tag{23}$$

$$U_2(\xi) = B_0 + B_1\varphi(\xi) \tag{24}$$

and $\varphi(\xi)$ satisfies the Riccati equation

$$\varphi'(\xi) = f + \varphi^2(\xi) \tag{25}$$

where f real-valued constant that is independent on ξ. Eq. (25) is the well known Riccati equation, which admits the following explicit solutions:

$$\varphi(\xi) = \sqrt{f}\ \tan(\sqrt{f}\,\xi) \tag{26}$$

$$\varphi(\xi) = -\sqrt{f}\ \cot(\sqrt{f}\,\xi) \tag{27}$$

when $f > 0$, and

$$\varphi(\xi) = -\sqrt{-f}\ \tanh(\sqrt{-f}\,\xi) \tag{28}$$

$$\varphi(\xi) = -\sqrt{-f}\ \coth(\sqrt{-f}\,\xi) \tag{29}$$

when $f < 0$, and

$$\varphi(\xi) = -\frac{1}{\xi} \tag{30}$$

when $f = 0$.

Substituting Eqs. (23) and (24) along with (25) in Eqs. (21) and (22) leads to

$$\begin{aligned}&B^2(a_1 - b_1 v)(2A_1 f\varphi + 2A_1\varphi^3) \\ &+ \left(-\omega_1 - a_1\kappa_1^2 + b_1\kappa_1\omega_1 + \alpha_1\kappa_1\right)(A_0 + A_1\varphi) \\ &+ (\lambda_1\kappa_1 + \theta_1\kappa_1 + c_1)(A_0 + A_1\varphi)^3 \\ &+ d_1(B_0 + B_1\varphi)^2(A_0 + A_1\varphi) = 0\end{aligned} \tag{31}$$

$$\begin{aligned}&B^2(a_2 - b_2 v)(2B_1 f\varphi + 2B_1\varphi^3) \\ &+ \left(-\omega_2 - a_2\kappa_2^2 + b_2\kappa_2\omega_2 + \alpha_2\kappa_2\right)(B_0 + B_1\varphi) \\ &+ (\lambda_2\kappa_2 + \theta_2\kappa_2 + c_2)(B_0 + B_1\varphi)^3 \\ &+ d_2(A_0 + A_1\varphi)^2(B_0 + B_1\varphi) = 0\end{aligned} \tag{32}$$

Then, equating the coefficient of each power of $\varphi(\xi)$ to zero, we obtain a system of nonlinear algebraic equations which solve to

$$\omega_1 = \frac{\begin{pmatrix} A_1^2 f(\lambda_1\kappa_1 + \theta_1\kappa_1 + c_1) \\ + fd_1B_1^2 + a_1\kappa_1^2 - \alpha_1\kappa_1 \end{pmatrix}}{b_1\kappa_1 - 1} \tag{33}$$

$$\omega_2 = \frac{\begin{pmatrix} B_1^2 f(\lambda_2\kappa_2 + \theta_2\kappa_2 + c_2) \\ + fd_2A_1^2 + a_2\kappa_2^2 - \alpha_2\kappa_2 \end{pmatrix}}{b_2\kappa_2 - 1} \tag{34}$$

$$B = \left[\frac{\begin{pmatrix} A_1^2\left(-b_2c_1 - b_2\lambda_1\kappa_1 - b_2\theta_1\kappa_1 + d_2b_1\right) \\ + B_1^2\left(\lambda_2\kappa_2b_1 + \theta_2\kappa_2b_1 + c_2b_1 - b_2d_1\right) \end{pmatrix}}{2(b_2a_1 - a_2b_1)}\right] \tag{35}$$

$$v = \frac{\begin{pmatrix} B_1^2\{(\lambda_2\kappa_2 + \theta_2\kappa_2 + c_2)a_1 - a_2d_1\} \\ + A_1^2\{-(c_1 + \lambda_1\kappa_1 + \theta_1\kappa_1)a_2 + d_2a_1\} \end{pmatrix}}{\begin{pmatrix} A_1^2\left(d_2b_1 - b_2\left(c_1 + \lambda_1\kappa_1 + \theta_1\kappa_1\right)\right) \\ + B_1^2\left(b_1\left(\lambda_2\kappa_2 + \theta_2\kappa_2 + c_2\right) - b_2d_1\right) \end{pmatrix}} \tag{36}$$

$$A_1 = A_1,\ B_1 = B_1,\ A_0 = B_0 = 0 \tag{37}$$

Now, equating the two values of v from (18) and (36) leads to

$$
\begin{aligned}
&(b_l\kappa_l - 1)\left[\begin{array}{l} B_1^2\left\{\begin{array}{l}(\lambda_2\kappa_2 + \theta_2\kappa_2 + c_2)a_1 \\ -a_2 d_1\end{array}\right\} \\ +A_1^2\left\{\begin{array}{l}-(c_1 + \lambda_1\kappa_1 + \theta_1\kappa_1)a_2 \\ +d_2 a_1\end{array}\right\}\end{array}\right] \\
&= (2\kappa_l a_l - b_l\omega_l - \alpha_l) \\
&\times\left[\begin{array}{l} A_1^2\left(d_2 b_1 - b_2(c_1 + \lambda_1\kappa_1 + \theta_1\kappa_1)\right) \\ +B_1^2\left(-b_2 d_1 + b_1\left(\begin{array}{l}\lambda_2\kappa_2 + \theta_2\kappa_2 \\ +c_2\end{array}\right)\right)\end{array}\right]
\end{aligned}
\tag{38}
$$

Finally, using solutions (26)-(30) of Eq. (25), we obtain the the following singular periodic solutions for the two components

$$
q(x,t) = A_1\sqrt{f}\,\tan\left(\sqrt{f}B(x - vt)\right) \times e^{i(-\kappa_1 x + \omega_1 t + \sigma_1)} \tag{39}
$$

and

$$
r(x,t) = B_1\sqrt{f}\,\tan\left(\sqrt{f}B(x - vt)\right) \times e^{i(-\kappa_2 x + \omega_2 t + \sigma_2)} \tag{40}
$$

as well as

$$
q(x,t) = -A_1\sqrt{f}\,\cot\left(\sqrt{f}B(x - vt)\right) \times e^{i(-\kappa_1 x + \omega_1 t + \sigma_1)} \tag{41}
$$

and

$$
r(x,t) = -B_1\sqrt{f}\,\cot\left(\sqrt{f}B(x - vt)\right) \times e^{i(-\kappa_2 x + \omega_2 t + \sigma_2)} \tag{42}
$$

These solutions exist for $f > 0$. However, for $f < 0$, one recovers the dark 1-soliton solution given by

$$
q(x,t) = -A_1\sqrt{-f}\,\tanh\left(\sqrt{-f}B(x - vt)\right) \times e^{i(-\kappa_1 x + \omega_1 t + \sigma_1)} \tag{43}
$$

and

$$
r(x,t) = -B_1\sqrt{-f}\,\tanh\left(\sqrt{-f}B(x - vt)\right) \times e^{i(-\kappa_2 x + \omega_2 t + \sigma_2)} \tag{44}
$$

or singular 1-soliton solutions given by

$$
q(x,t) = -A_1\sqrt{-f}\,\coth\left(\sqrt{-f}B(x - vt)\right) \times e^{i(-\kappa_1 x + \omega_1 t + \sigma_1)} \tag{45}
$$

and

$$
r(x,t) = -B_1\sqrt{-f}\,\coth\left(\sqrt{-f}B(x - vt)\right) \times e^{i(-\kappa_2 x + \omega_2 t + \sigma_2)} \tag{46}
$$

Finally, for $f = 0$, the following rational solutions emerge:

$$
q(x,t) = -\frac{A_1}{B(x - vt)}e^{i(-\kappa_1 x + \omega_1 t + \sigma_1)} \tag{47}
$$

and

$$
r(x,t) = -\frac{B_1}{B(x - vt)}e^{i(-\kappa_2 x + \omega_2 t + \sigma_2)} \tag{48}
$$

where B is given by Eq. (35), while v is given by Eq. (36), and ω_l are given by Eqs. (33) and (34). These solutions will be defined subjected to the constraint conditions (13)-(15) and (19)-(20) and (38).

4. Conclusion

This paper recovered dark and singular optical solitons in birefringent fibers where in addition to several perturbation terms STD is considered along with GVD for well-posedness of the governing model. Riccati equation method retrieved soliton as well as other solutions to the model. These additional solutions are singular periodic solutions and plane waves. Although these non-soliton solutions are not applicable in optics, they are listed to obtain a complete spectrum of solutions. It is unfortunate that this integration algorithm cannot retrieve soliton solutions for birefringent fibers in parabolic law medium, this paper therefore was confined to Kerr law nonlinearity. Later, this integration will be further applied to several additional situations such as cascaded systems, Thirring solitons, DWDM systems and optical couplers, just to name a few. The results of those research will be disclosed in future.

Acknowledgment

This research is funded by Qatar National Research Fund (QNRF) under the grant number NPRP 6-021-1-005. The fourth and fifth authors (AB & MB) thankfully acknowledge this support from QNRF.

References

[1] M. Bache, F. W. Wise, Physical Review A **81**, 053815 (2010).

[2] A. H. Bhrawy, A. A. Alshaery, E. M. Hilal, Michelle Savescu, Daniela Milovic, Kaisar R. Khan, Mohammad F. Mahmood, Zlatko Jovanoski, A. Biswas, Optik **125**, 4935 (2014).

[3] A. H. Bhrawy, A. A. Alshaery, E. M. Hilal, Z. Jovanoski, A. Biswas, Optik **125**, 6162 (2014).

[4] A. Biswas, K. Khan, A. Rahman, A. Yildirim, T. Hayat, O. M. Aldossary, J. Optoelectron. Adv. Mater. **14**, 571 (2012).

[5] J-V. Guzman, E. M. Hilal, A. A. Alshaery, A. H. Bhrawy, M. F. Mahmood, L. Moraru, A. Biswas, To appear in Proceedings of the Romanian Academy, Series A.

[6] S. Kumar, K. Singh, R. K. Gupta, Pramana **79**, 41 (2012).

[7] H. Li, D. Wang, Chinese Optics Letters **5**, 504 (2007).

[8] A. F. Mohammed, C. H. Telk, S. P. Majumdar, Journal of Optical Communications **21**, 165 (2000).

[9] A. K. Sarma, A. Biswas, Optica Applicata **44**, 205 (2014).

[10] M. Savescu, A. H. Bhrawy, E. M. Hilal, A. A. Alshaery, A. Biswas, Romanian Journal of Physics **59**, 582 (2014).

[11] W. P. Zhong, Communications in Theoretical Physics **33**, 157 (2000).

[12] Q. Zhou, D. Yao, F. Chen, W. Li, Journal of Modern Optics **60**, 854 (2013).

[13] Q. Zhou, D. Yao, F. Chen, Journal of Modern Optics **60**, 1652 (2013).

[14] Q. Zhou, D. Yao, X. Liu, F. Chen. S. Ding, Y. Zhang, F. Chen, Optics and Laser Technology **51**, 32 (2013).

[15] Q. Zhou, Optik **125**, 3142 (2014).

[16] Q. Zhou, Q. Zhu, A. Biswas, Optica Applicata **44**, 399 (2014).

[17] Q. Zhou, Q. Zhu, A.H. Bhrawy, L. Moraru, A. Biswas, Optoelectronics and Advanced Materials: Rapid Communications **8**, 800 (2014).

[18] Q. Zhou, Journal of Modern Optics **61**, 500 (2014).

[19] Q. Zhou, Q. Zhu, Waves in Random and Complex Media http://dx.doi.org/10.1080/17455030.2014.956847

[20] Q. Zhou, Q. Zhu, Y. Liu, H. Yu, P. Yao, A. Biswas, Laser Physics 25, 014502. (2015).

*Corresponding author: biswas.anjan@gmail.com