Non-existence of Shilnikov chaos in a simple five-term chaotic system with exponential quadratic term

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This paper investigates a simplified five-term chaotic system with exponential quadratic term by detailed theoretical analysis as well as dynamic simulation, including some basic dynamical properties, Lyapunov exponent spectra, Poincaré mapping, fractal dimension, bifurcation diagram, routes to chaos, and forming mechanisms of its compound structures. The obtained results show clearly that the system with two non-hyperbolic equilibria for all $a > 0$ and $b > 1$ deserves a further detailed investigation.

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1. Introduction

Since the discovery of the eminent Lorenz chaotic attractor in 1963 [1], the study of chaotic behavior in nonlinear systems has attracted great attention due to many of possible applications in various fields of science and technology. Subsequently, the system has been extensively studied with many important results in chaotic dynamics, control, and synchronization. In 1999, Chen constructed a 3-D chaotic system [2] by a simple state feedback to the second equation in the Lorenz system that combines features of both the Lorenz attractor and the Rössler attractor [3]. In 2002, Lü and Chen investigated another a new chaotic attractor connecting the Lorenz attractor and Chen's attractor [4]. Later, many Lorenz-like or Lorenz-based chaotic systems were proposed and investigated. Sprott embarked upon an extensive search for autonomous three-dimensional chaotic systems with fewer than seven terms in the right hand side of the model equations. He considered general three dimensional ordinary differential equations with quadratic nonlinearities and found by computer simulation 19 simple 3-D quadratic autonomous chaotic systems with none, one equilibrium or two equilibria [5-7]. Moreover, chaotic generator research and design studies have become a pivotal point for many electronics engineers [8-11].

It is noted that some classical 3-D autonomous chaotic systems have three particular fixed points: one saddle and two unstable saddle-foci [1, 2-4]. The other 3D chaotic systems have two unstable saddle-foci [2]. In 2008, Yang and Chen found another 3-D chaotic system with three fixed points: one saddle and two stable equilibria [12]. In 2010, an unusual 3D autonomous quadratic Lorenz-like chaotic system with only two stable node-foci was proposed by Yang, Wei and Chen [13]. In 2011, Wang and Chen obtained chaotic attractors with only one stable node-focus by adding a simple constant control parameter to Sprott E system [5]. Recently, a chaotic system with no equilibria was proposed by Wei [14], which was illustrated in the case of a period-doubling sequence of bifurcations leading to a Feigenbaum-like strange attractor.

In this paper, a new simple chaotic system with a total of five terms on the right hand side, of which there are only one quadratic nonlinear term and one exponential quadratic nonlinear term, is proposed in three first-order autonomous ODEs. Some basic dynamical behaviors are further explored by calculating its Lyapunov exponent spectra and bifurcation diagrams. Such a new attractor not only contributes an addition to the rarely-found five-term chaotic systems, but also in some sense is simpler than those of existing seven-term or six-term equations. Therefore, this would be of mathematical and practical interests.

2. The simplified five-term system

2.1. Chaotic attractor

The new three-dimensional chaotic system is constructed by introducing an exponential quadratic term in three-dimensional Lorenz-like equations, which is described by

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -xz \\ \dot{z} = -b + e^{y^2} \end{cases} \tag{1}$$

where a and b is a real constant, and x, y, z are the state variables.

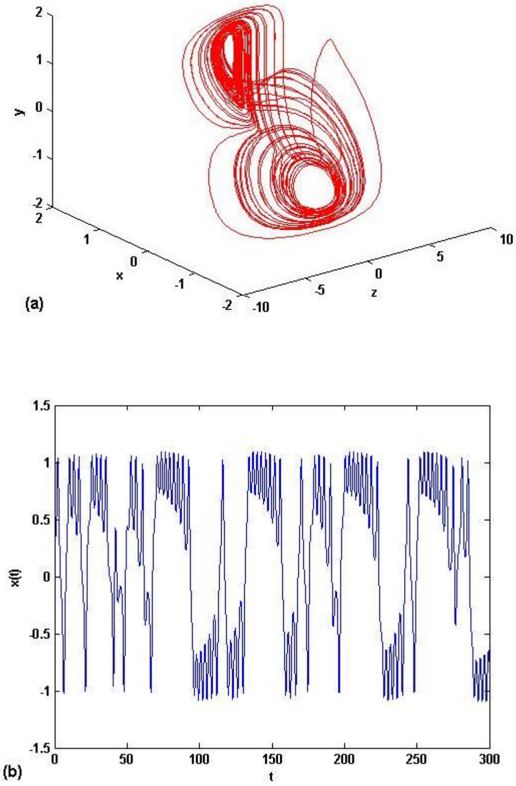

Fig. 1. (a) The chaotic attractor of system (1) for c = 2; (b) Waveform of the state variable x(t).

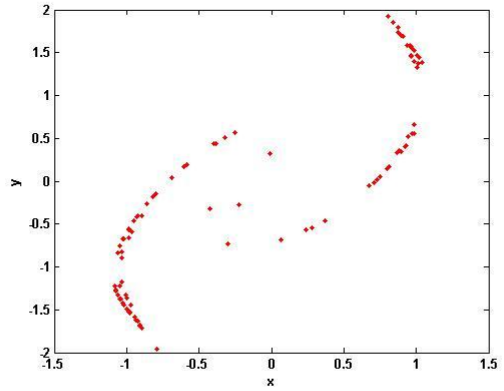

Fig. 2. Poincaré mapping on the z=1 section.

It is easy to verity the system (1) and all other existing three dimensional quadratic systems, such as Lorenz system [1], Chen system [2], Lü system [4], are not topologically equivalent since the former have an exponential quadratic term and two equilibria. Therefore it is straightforward to verify that there is no non-singular coordinate transforms that can convert such system to other existing chaotic system.

2.2. Some basic properties

System (1) has several additional important properties:

(1) Symmetry and invariance

System (1) is symmetric and invariant under the transformation $(x, y, z) \to (-x, -y, z)$, i.e., reflection about the z-axis. Also, the z-axis itself is an orbit (an invariant manifold), i.e., if $x = y = 0$ at $t = t_0$ then $x = y = 0$ for all $t \geq t_0$ and $b > 1$. Furthermore, the trajectory on the z-axis tends to the origin as $t \to -\infty$, since for such a trajectory, $\dot{x} = \dot{y} = 0$ and $\dot{z} = -b + 1$. Therefore, system (1) has this symmetry and invariance.

(2) Dissipation and the existence of attractor

The rate of volume contraction is given by the Lie derivative

$$\frac{1}{V}\frac{dV}{dt} = \sum_i \frac{\partial \dot{\phi}_i}{\partial \phi_i}, i = 1,2,3, \quad \phi_1 = x, \quad \phi_2 = y, \quad \phi_3 = z \quad (2)$$

For dynamical system (1), we obtain

$$\frac{1}{V}\frac{dV}{dt} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a \quad (3)$$

which can be solved to yield

$$V(t) = V(0)e^{-at} \quad (4)$$

For $a > 0$, and the dynamical system (1) is dissipative with solutions for $t \to \infty$ that contract at an exponential rate $-a$ onto an attractor of zero volume that may be an equilibrium point, a limit cycle, or a strange attractor.

(3) Equilibria and stability

The equilibria of system (1) can be found by solving the three equations $\dot{x} = \dot{y} = \dot{z} = 0$, which lead to

$$a(y - x) = 0, -xz = 0, \text{ and } -b + e^{y^2} = 0.$$

When $a > 0$ and $b > 1$, there are two equilibria: $S_-(-\sqrt{\ln b}, -\sqrt{\ln b}, 0)$ and $S_+(\sqrt{\ln b}, \sqrt{\ln b}, 0)$, in which S_- and S_+ are symmetrically placed with respect to the z-axis.

Next, linearizing the system about t the equilibrium S_- yields the following characteristic equation:

$$f(\lambda) = \lambda^3 + a\lambda^2 + 2b\ln b\lambda + 2ab\ln b \quad (6)$$

These two equilibria $S_\pm$ have the same stability characterization. The characteristic roots are $\lambda_1 = -a$, $\lambda_{2,3} = \pm i\sqrt{b\ln b}$, thus two equilibria are non-hyperbolic. Therefore, Hopf bifurcation can not appear at the two equilibria.

3. Dynamical behavior of the system when *a*=1

3.1 The Lyapunov exponent spectrum and routes to Chaos

As it is well known, the Lyapunov exponents measure the exponential rates of divergence and convergence of nearby trajectories in state space, and the Lyapunov exponent spectrum provides additional useful information about the system. The two largest Lyapunov exponents of are shown in Fig. 3. A positive and zero Lyapunov exponent indicates chaos, two zero Lyapunov exponents indicate a bifurcation, and a zero and a negative Lyapunov exponent indicates periodicity (a limit cycle).

The range of dynamical behaviors is shown by the bifurcation diagram in Fig. 4 in which successive values of $x_{\max}$ are plotted at each value of *b*. The band structure indicates chaos, which disappears as *b* increases.

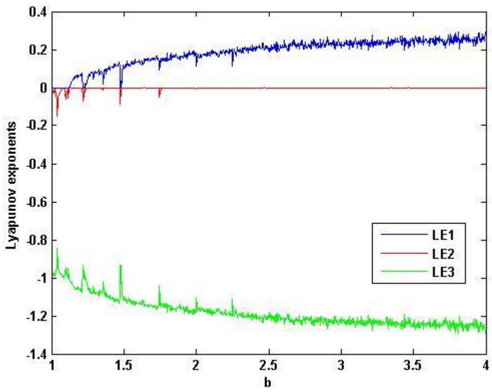

Fig. 3. The Lyapunov exponents (blue, green and red, respectively) versus $b \in (1,4]$ (Time step: 0.01, Initial condition: (0.5, 0.06, -0.49), Iterations: 800000).

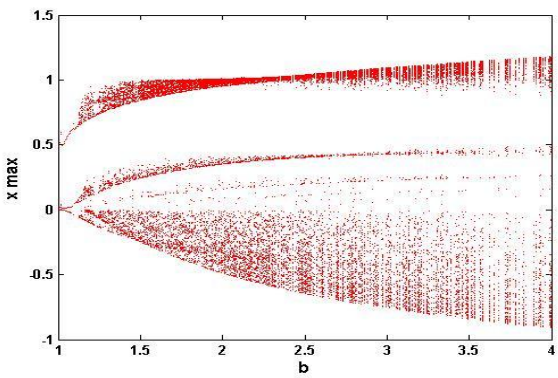

Fig. 4. The bifurcation diagram of x_{max} versus b.

Note that system (1) is chaotic over most of the range $b \in (1,4]$ with some windows of periodicity in the range $b \in (1,2]$, such as $W_1 = (1,1.123]$, $W_2 = [1.212,1.232]$ and $W_3 = [1.475,1.48)$. Different windows exhibit different periodic obits. Some of these periodic orbits projected onto the *xz*-plane with different values of *c* are shown in Fig. 5. For application to secure communication, one should avoid these windows.

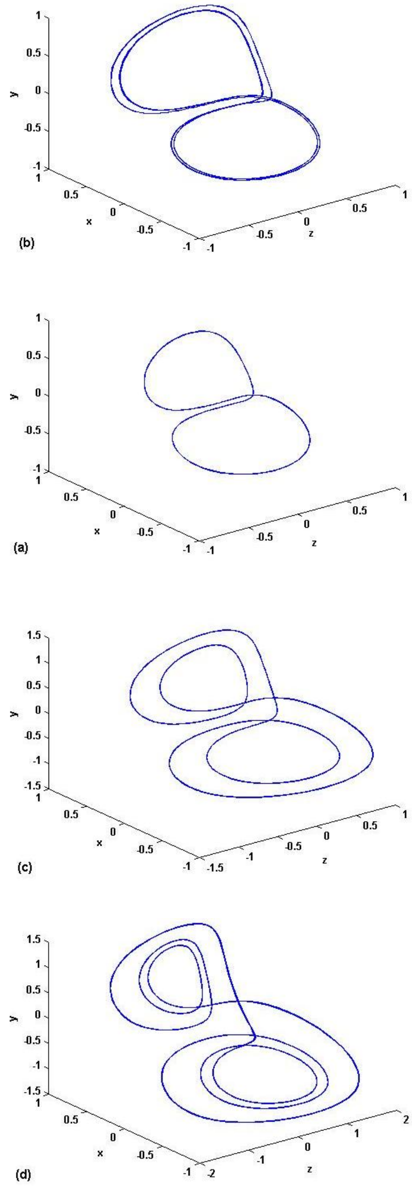

Fig. 5. Periodic orbits of system (1) for different windows of periodicity with b (a) b=1.07 (b) b = 1.12 (c) b = 1.22 (d) b = 1.476.

3.2 The Kaplan-Yorke dimension

Whereas the Lyapunov exponent measures the average predictability of a dynamical system, the dimension of its attractor measures its complexity. A fractional dimension can be defined:

$$D_{KY} = D + \frac{1}{|\lambda_{D+1}|}\sum_{j=1}^{D}\lambda_j \tag{7}$$

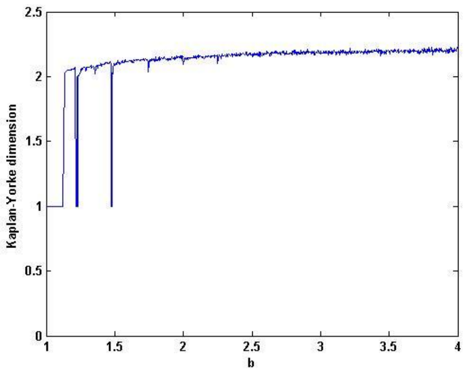

Fig. 6. The Kaplan-Yorke dimension of system (1).

The Kaplan-Yorke dimension of system (1) is shown in Fig. 6. The dimension of system (1) is larger than 2 for a strange attractor, and is 1.0 for a limit cycle. The system has no hyperbolic equilibria over the range $b \in (1,4]$.

4. Non-existence of Shilnikov chaos in system (1)

Let us consider the 3th-order autonomous system

$$\frac{dx}{dt} = f(x), \tag{8}$$

where the vector field $f(x) = (f_1, f_2, f_3)^T : R^3 \to R^3$: belongs to the class $C^r (r \geq 1)$, $x = (x_1, x_2, x_3)^T$ is the state variable of the system, and $t \in R$ is the time. Suppose that $f(x)$ has at least one equilibrium point P.

According to the **Theorem 1** in [15], we have the following result.

Theorem 4.1 System (1) can not have homoclinic and heteroclinic orbits.

Proof First, let P be an equilibrium point of system (1). Then, if there exists a homoclinic orbit

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))^T,$$

$$\lim_{t \to +\infty} \gamma(t) = \lim_{t \to -\infty} \gamma(t) = P = (p_1, p_2, p_3).$$

Because of $x_3(t) \geq -b$, a simple integration from t_0 to t that

$$x_3(t) \geq -b(t - t_0) + x_3(t_0), \tag{9}$$

where t_0 is the initial time such that $t \geq t_0$, thus, using (9) one has $\gamma_3(t) \geq -b(t - t_0) + \gamma_3(t_0)$, and $\lim_{t \to -\infty} \gamma_3(t) = +\infty \neq p_3$, that is, at least one component of $\gamma(t)$ is not bounded.

Now, let P_1 and P_2 be saddle foci of system (1). If there exists a heteroclinic orbit $\delta(t) = (\delta_1(t), \delta_2(t), \delta_3(t))$

Then, from inequality (9), one has $\lim_{t \to -\infty} \delta_3(t) = +\infty$.

Thus, at least one component of $\delta(t)$ is not bounded. Therefore, the system (1) has no nomoclinic and heteroclinic orbits. From Theorem 4.1, it is important to remark that if system (1) is chaotic, then its chaos is not of the horseshoe type.

5. Conclusions

A new five-term simple chaotic attractor has been proposed and characterized by five terms in three first-order autonomous ODEs. In terms of algebraic representation, such a five-term attractor is of particular interest as it is in some sense simpler than other existing seven-term or six-term attractors, and displays a novel 2-scroll chaotic attractor with two non-hyperbolic equilibria. Some basic properties of the new system have been investigated in terms of chaotic attractors, equilibria, eigenvalues of the Jacobian matrices, Lyapunov exponents, a dissipative system or an existence of the attractor, Poincaré maps, bifurcations. There are additional interesting features of this system with the non-existence of homoclinic and heteroclinic orbits in terms of control, synchronization, circuit implementation and its application to secure communications that deserve further study.

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