Instability result of a fifth order non-linear delay system

CEMİL TUNÇ
Department of Mathematics, Faculty of Sciences, Yüzüncü Yıl University, 65080, Van -Turkey

This paper considers a fifth order nonlinear differential equation with a constant delay. Some sufficient conditions for the instability of the zero solution are established, which are new and complement previously known results.

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1. Introduction

In the last few years, the instability of solutions for fifth order differential equations without delay has received a lot of attention. Some of these results can be found in Ezeilo [2-4], Li and Duan [7], Li and Yu [8], Sadek [9], Sun and Hou [10], Tiryaki [11], Tunç [12-14], Tunç and Erdogan [17], Tunç and Karta [18] and Tunç and Şevli [19]. However, to the best of our knowledge, the author in ([15], [16]) has only considered the instability of the solutions of some fifth order nonlinear differential equations with delay. Thus, it is worthwhile to continue to investigate the instability of the solutions of fifth order delay differential equations in this case. In regard to fifth order nonlinear delay differential equations, in 2011, Tunç ([15], [16]) discussed the instability of the zero solution of the differential equations

$$x^{(5)} + \psi_1(x'')x''' + \phi(x, x(t-r), ..., x^{(4)}, x^{(4)}(t-r))x'' + \theta_1(x') + f_1(x(t-r)) = 0$$

and

$$x^{(5)} + a_1 x^{(4)} + k(x, x', x'', x''', x^{(4)})x''' + g(x')x'' + h(x, x', x'', x'', x^{(4)}) + f(x(t-r)) = 0,$$

respectively.

Meanwhile, in 2000, Li and Duan [7] established an instability theorem for the fifth order nonlinear differential equation without delay

$$x^{(5)}(t) + f_5(x(t), x'(t), x''(t), x'''(t), x^{(4)}(t))x^{(4)}(t) + f_4(x(t), x'(t), x''(t), x'''(t), x^{(4)}(t))x'''(t) + f_3(x''(t)) + f_2(x'(t)) + a_1 x(t) = 0. \tag{1}$$

In this paper, instead of Eq. (1), we consider fifth order nonlinear delay differential equation

$$x^{(5)}(t) + f_5(x(t-r), x'(t-r), x''(t-r), x'''(t-r), x^{(4)}(t-r))x^{(4)}(t) + f_4(x(t-r), x'(t-r), x''(t-r), x'''(t-r), x^{(4)}(t-r))x'''(t) + f_3(x''(t-r)) + f_2(x'(t)) + a_1 x(t) = 0. \tag{2}$$

We write Eq. (2) as the system

$$x_1' = x_2,\ x_2' = x_3,\ x_3' = x_4,\ x_4' = x_5,\ x_5' = - f_5(x_1(t-r), x_2(t-r), x_3(t-r), x_4(t-r), x_5(t-r))x_5 - f_4(x_1(t-r), x_2(t-r), x_3(t-r), x_4(t-r), x_5(t-r))x_4 - f_3(x_3) - f_2(x_2) - a_1 x_1 + \int_{t-r}^{t} f_3'(x_3(s))x_4(s)ds, \tag{3}$$

where $a_1\ (< 0)$ and $r\ (> 0)$ are constants, r is fixed delay, the primes in Eq. (2) denote differentiation with respect to $t, t \in \Re^+ = [0, \infty)$; f_2, f_3, f_4 and f_5 are continuous functions on $\Re, \Re, \Re^5$ and $\Re^5$, respectively, and with $f_2(0) = f_3(0) = 0$. The continuity of the functions f_2, f_3, f_4 and f_5 is a sufficient condition for the existence of the solution of Eq. (2) (see [1, pp. 14]). It is also assumed as basic that the functions f_2, f_3, f_4 and f_5 satisfy a Lipschitz condition in their respective arguments. Hence, the uniqueness of solutions of Eq. (2) is guaranteed (see [1, pp.15]). We also assume in what follows that f_3 is differentiable, and $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $x_5(t)$ are abbreviated as x_1, x_2, x_3, x_4 and x_5, respectively.

The purpose of this paper is to present a new result on the instability of the zero solution of Eq. (2). Our method relies on the the Lyapunov-Krasovskii functional approach (see [5]). This method permits us to obtain new result on Eq. (2) under quite general assumptions on the nonlinearities. The obtained result improves and enhances the result in Li and Duan [7, Theorem 5] for the case without delay to the case with delay. Here, by defining an appropriate Lyapunov functional, we carry out our purpose.

In the following theorems, we give basic idea of the method about the instability of solutions of ordinary and delay differential equations. The following theorem, due to the Russian mathematician N. G. Četaev's (see LaSalle and Lefschetz [6]).

Theorem A (Instability Theorem of Četaev's)**.** Let Ω be a neighborhood of the origin. Let there be given a function $V(x)$ and region Ω_1 in Ω with the following properties:

(i) $V(x)$ has continuous first partial derivatives in Ω_1.

(ii) $V(x)$ and $\dot{V}(x)$ are positive in Ω_1.

(iii) At the boundary points of Ω_1 inside Ω, $V(x)=0$.

(iv) The origin is a boundary point of Ω_1.

Under these conditions the origin is unstable.

Let $r \geq 0$ be given, and let $C = C([-r,0],\ \Re^n)$ with

$$\|\phi\| = \max_{-r\leq s\leq 0}|\phi(s)|, \phi \in C.$$

For $H>0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x:[-r,a]\to\Re^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t+s), -r \leq s \leq 0,\ t \geq 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t),\ x_t = x(t+\theta),\ -r \leq \theta \leq 0,\ t \geq 0,$$

where $F: G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from the conditions on F that each initial value problem

$$\dot{x} = F(x_t),\ x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A), 0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi.$

Definition. The zero solution, $x=0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

2. Main results

Our main result is the following theorem.

Theorem. Assume that there exist positive constants a_3 and δ such that the following conditions hold:

$$f_2(0)=0,\ f_2(x_2)\neq 0,\ (x_2\neq 0),\ f_3(0)=0,$$
$$f_3(x_3)\neq 0,\ (x_3\neq 0),$$

$f_2'(x_2) \geq 0$ for all x_2, $-a_3 \leq f_3'(x_3) \leq a_3$ for all x_3 and

$$f_4(x_1(t-r),...,x_5(t-r)) + \frac{1}{4} f_5^2(x_1(t-r),...,x_5(t-r)) \leq -\delta$$

for all $x_1(t-r),..., x_5(t-r)$.

Then, the zero solution of Eq. (2) is unstable.

It should be noted that the proof of the main result is based on the instability criteria of Krasovskiĭ [5]. Because of these criteria, it is necessary to show here that there exists a Lyapunov functional $V = V(x_{1t},...,x_{5t})$ which has Krasovskiĭ properties, say (P_1), (P_2) and (P_3):

(P_1) In every neighborhood of $(0,0,0,0,0)$, there exists a point $(\xi_1,...,\xi_5)$ such that $V(\xi_1,...,\xi_5) > 0$,

(P_2) the time derivative $\frac{d}{dt}V(x_{1t},...,x_{5t})$ along solution paths of (3) is positive semi-definite,

(P_3) the only solution $(x_1,...,x_5) = (x_1(t),...,x_5(t))$ of (3) which satisfies $\frac{d}{dt}V(x_{1t},...,x_{5t}) = 0, (t>0)$, is the trivial solution $(0,0,0,0,0)$.

Proof. We define a Lyapunov functional $V = V(x_{1t},...,x_{5t})$:

$$V = x_4x_5 - \int_0^{x_3} f_3(s)ds + f_2(x_2)x_3 + a_1x_1x_3 - \frac{1}{2}a_1x_2^2$$
$$-\lambda_1 \int_{-r}^{0}\int_{t+s}^{t} x_4^2(\theta)d\theta ds, \tag{4}$$

where s is a real variable such that the integral $\int_{-r}^{0}\int_{t+s}^{t} x_4^2(\theta)d\theta ds$ is non-negative, and λ_1 is positive constant which will be determined later in the proof.

Hence, it is clear from the definition of V that

$$V(0,0,0,0,0) = 0$$

and

$$V(0,\varepsilon,0,0,0) = -\frac{1}{2}a_1\varepsilon^2 > 0$$

for all sufficiently arbitrary small ε so that every neighborhood of the origin in the $(x_1,...,x_5)-$ space contains points $(\xi_1,...,\xi_5)$ such that $V(\xi_1,...,\xi_5) > 0.$

Let

$$(x_1,...,x_5) = (x_1(t),...,x_5(t))$$

be an arbitrary solution of (3). By an elementary differentiation, time derivative of the Lyapunov functional V in (4) along the solutions of (3) yields

$$\frac{d}{dt}V(x_{1t},...,x_{5t})$$
$$= x_5^2 - f_5(x_1(t-r),...,x_5(t-r))x_4x_5 - f_4(x_1(t-r),...,x_5(t-r))x_4^2$$
$$+ f_2'(x_2)x_3^2$$
$$+ x_4\int_{t-r}^{t} f_3'(x_3(s))x_4(s)ds - \lambda_1 r x_4^2 + \lambda_1\int_{t-r}^{t} x_4^2(s)ds$$

$$= \left(x_5 - \frac{1}{2}f_5(x_1(t-r),...,x_5(t-r))x_4\right)^2$$
$$-\{f_4(x_1(t-r),...,x_5(t-r)) + \frac{1}{4}f_5^2(x_1(t-r),...,x_5(t-r))\}x_4^2$$

$$+ f_2'(x_2)x_3^2 + x_4\int_{t-r}^{t} f_3'(x_3(s))x_4(s)ds$$
$$-\lambda_1 r x_4^2 + \lambda_1\int_{t-r}^{t} x_4^2(s)ds$$

$$\geq \delta x_4^2 + f_2'(x_2)x_3^2 + x_4\int_{t-r}^{t} f_3'(x_3(s))x_4(s)ds$$
$$-\lambda_1 r x_4^2 + \lambda_1\int_{t-r}^{t} x_4^2(s)ds.$$

Making use of the assumption $-a_3 \leq f_3'(x_3) \leq a_3$ and the estimate $2|ab| \leq a^2 + b^2$, we have

$$x_4\int_{t-r}^{t} f_3'(x_3(s))x_4(s)ds \geq -|x_4|\int_{t-r}^{t}|f_3'(x_3(s))|\,|x_4(s)|ds$$
$$\geq -\frac{1}{2}a_3 r x_4^2 - \frac{1}{2}a_3\int_{t-r}^{t} x_4^2(s)ds.$$

Then

$$\frac{d}{dt}V(x_{1t},...,x_{5t}) \geq \{\delta - (a_3+\lambda_1)r\}x_4^2 + f_2'(x_2)x_3^2$$
$$+\left(\lambda_1 - \frac{1}{2}a_3\right)\int_{t-r}^{t} x_4^2(s)ds.$$

Let $\lambda_1 = \frac{1}{2}a_3$ and $r < \frac{2\delta}{3a_3}$ so that

$$\frac{d}{dt}V(x_{1t},...,x_{5t}) \geq f_2'(x_2)x_3^2 + \alpha x_4^2 > 0$$

for a positive constant α. Thus, the Lyapunov functional V satisfies the property (P_2).

Besides, $\frac{d}{dt}V(x_{1t},...,x_{5t}) = 0$ if and only if

$$x_3 = x_4 = x_5 = 0. \qquad (5)$$

The substitution (5) in (2) implies

$$f_2(x_2) + a_1x_1 = 0.$$

That is,

$$f_2(x_1') + a_1x_1 = 0. \qquad (6)$$

Because, $x_1'' = 0,\ x_1' = \text{constant, for all}\, t > 0.$ Hence, by (6), since $a_1 \neq 0,$ we have that $x_1 = \text{constant, for all}\, t > 0.$ But this implies that $x_1' = 0$ and thus also, by (6), that $x_1 = 0,\ \text{for all}\, t > 0.$ These estimates imply that $x_1 = x_2 = x_3 = x_4 = x_5 = 0.$ Hence the property (P_3) holds for the Lyapunov functional V. The theorem is thereby established.

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*Corresponding author: cemtunc@yahoo.com