Improved combination of variational and effective index methods for optical rib and box-shaped waveguides

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We improve the values of the propagation constants for guided TE_0 and TM_0 modes in optical rib and box shaped waveguides by using a combination of variational and effective index methods.

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1. Introduction

The knowledge of optical propagation characteristics of the optical rib and box shaped waveguides is very important for the project of integrated optics [1-5]. The effective index method produces values for the propagation constant which are too high in comparison with the exact values [6]. In contrast, the values of the propagation constants in optimal variational methods are smaller than the exact values [1].

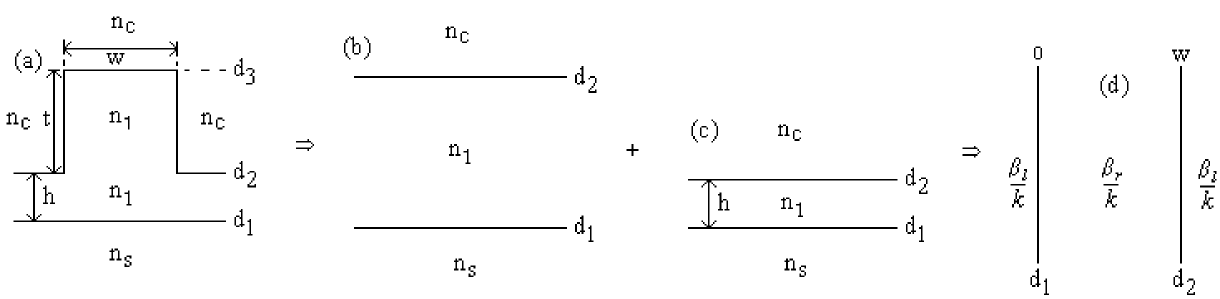

Fig. 1. Cross section of a typical rib waveguide and slab waveguide components.

In this paper, in the case of optical rib waveguides, we approximate the effective index of one of the outer slab regions with the variational effective index and the other one with the usual effective index from the effective index method. Thus, we transform a symmetric slab waveguide with three layers in one asymmetric. For a box shaped waveguide we obtain an asymmetric slab waveguide with five layers, where two of the lateral regions are approximated with the variational effective index and one exterior (VM-EIM$_1$) or two (VM-EIM$_2$) of the other lateral regions with the usual effective index.

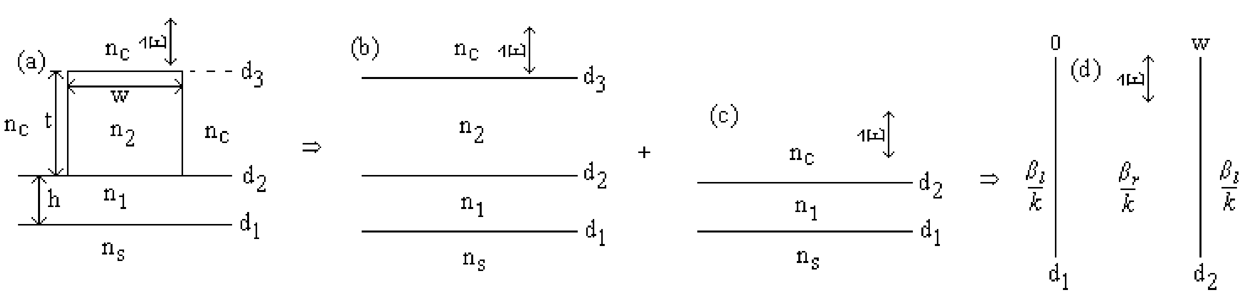

Fig. 2. Cross section of a typical dielectric loaded surface plasmon polariton rib waveguide and slab waveguide components.

2. Effective index method for optical rib and box shaped waveguides

The propagation constants β of an optical planar waveguide with multiple layers satisfy the dispersion equation [7]

$$F_N + \frac{\alpha_c}{n_c^{2\xi}} = 0\,, \tag{1}$$

where

$$F_1 = -\frac{\alpha_1}{n_1^{2\xi}}\tan[\alpha_1(d_2 - d_1) - \arctan[\frac{\alpha_s}{\alpha_1}(\frac{n_1}{n_s})^{2\xi}]]\,, \tag{2}$$

$$F_i = -\frac{\alpha_i}{n_i^{2\xi}}\tan[\alpha_i(d_{i+1} - d_i) - \arctan[\frac{F_{i-1}}{\alpha_i}n_i^{2\xi}]],\; i = 2,3,\ldots,N, \tag{3}$$

$$\alpha_s = \sqrt{\beta^2 - (n_s k)^2}\,,\; \alpha_i = \sqrt{(n_i k)^2 - \beta^2}\,,\; \alpha_c = \sqrt{\beta^2 - (n_c k)^2}\,,\; i = 1,2,3,\ldots,\mathrm{N} \tag{4}$$

ξ reads as 0 for TE polarized waves and 1 for TM polarized waves, N is the number of the intermediate layers between the substrate and cladding layers, k is the free space wave number and

$$n(x) = \begin{cases} n_s, & for \quad x < d_1 = 0, \\ n_i, & for \quad d_i < x < d_{i+1}, i = 1,2,\ldots,\mathrm{N}, \\ n_c, & for \quad d_{N+1} < x, \end{cases} \tag{5}$$

is the refractive index profile of the waveguide where n_s, n_c and n_i are the refractive index of the substrate, air cladding and intermediate layers, respectively.

Also, we can use the following variational method by using of the known exact analytical eigenfunctions. The variational solution of the scalar wave equation for a planar waveguide structure is found from the functional

$$J = \int_{-\infty}^{d_1} [-\frac{f_s'^{\,2}}{n_s^{2\xi}} + (k n_s^{1-\xi} f_s)^2] dx + \sum_{i=1}^{N} \int_{d_i}^{d_{i+1}} [-\frac{f_i'^{\,2}}{n_i^{2\xi}} + (k n_i^{1-\xi} f_i)^2] dx + \int_{d_{N+1}}^{\infty} [-\frac{f_c'^{\,2}}{n_c^{2\xi}} + (k n_c^{1-\xi} f_c)^2] dx\,, \tag{6}$$

subject to the constraint that

$$I = \int_{-\infty}^{d_1} \frac{f_s^2}{n_s^{2\xi}} dx + \sum_{i=1}^{N} \int_{d_i}^{d_{i+1}} \frac{f_i^2}{n_i^{2\xi}} dx + \int_{d_{N+1}}^{\infty} \frac{f_c^2}{n_c^{2\xi}} dx, \tag{7}$$

$$\beta^2 = \frac{J}{I} \tag{8}$$

where ξ reads as 0 for TE polarized waves and 1 for TM polarized waves and the exact functions fs, fi and fc are given by

$$f_s(x) = A_s \exp(\alpha_s x), \quad x \langle d_1\,, \tag{9}$$

$$f_i(x) = A_i \cos[\alpha_i(x - d_i)] + B_i \sin[\alpha_i(x - d_i)], \quad d_i < x < d_{i+1}, i = 1,2,3,\ldots,\mathrm{N}, \tag{10}$$

$$f_c(x) = A_c \exp(-\alpha_c(x - d_{N+1})), \quad x > d_{N+1}, \tag{11}$$

and

$$A_1 = A_s = 1, B_1 = \frac{n_1^{2\xi}}{n_s^{2\xi}}\frac{\alpha_s}{\alpha_1}, \tag{12}$$

$$A_i = A_{i-1}\cos[\alpha_{i-1}(d_i - d_{i-1})] + B_{i-1}\sin[\alpha_{i-1}(d_i - d_{i-1})], i = 2,3,\ldots,\mathrm{N}+1\,, \tag{13}$$

$$B_i = \frac{n_i^{2\xi}}{n_{i-1}^{2\xi}}\frac{\alpha_{i-1}}{\alpha_i}(B_{i-1}\cos[\alpha_{i-1}(d_i - d_{i-1})] - A_{i-1}\sin[\alpha_{i-1}(d_i - d_{i-1})]), i = 2,3,\ldots,\mathrm{N}+1\,, \tag{14}$$

$$A_c = A_{N+1}\cos[\alpha_N(d_{N+1} - d_N)] + B_{N+1}\sin[\alpha_N(d_{N+1} - d_N)]. \tag{15}$$

The integrals in equations (6 - 7) with the chosen exact functions are evaluated analytically to reduce the amount of numerical computation. The solutions of the dispersion equation (8)

$$\frac{J}{I} - \beta^2 = 0 \tag{16}$$

give the propagation constants β and the effective index β/k of the planar waveguide.

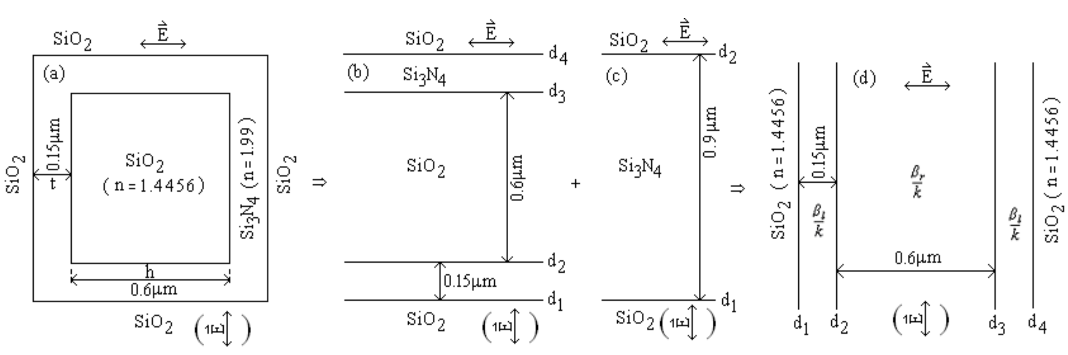

Fig. 3. Cross section of a typical box shaped waveguide and slab waveguide components.

These relations can be used to find the propagation constants in the effective index method of the rib and box shaped waveguides. For a rib waveguide, after finding the slab propagation constants β_r, β_l in the inner and outer regions of the waveguide (Fig. 1), these layered regions are replaced by the regions with effective indices β_r/k and β_l/k. This new symmetric slab waveguide with three layers is used to approximate the final mode effective index β/k of the rib waveguide. Each of the slab waveguide (Fig.1 (b)-(d)) can be solved for $N = 1$ in the relations (1-5) or (6-16).

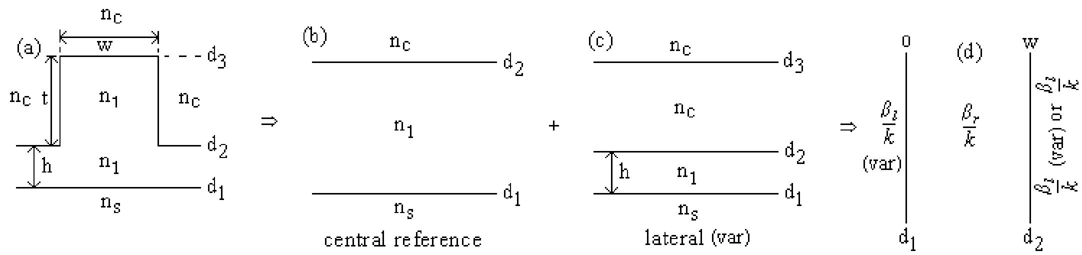

Fig. 4. Cross section of a rib waveguide and slab waveguide variational components.

For a dielectric loaded surface plasmon polariton rib waveguide (Fig.2) we follow the same procedure as above but in the first step (Fig.2 b) we use $N = 2$ in the relations (1-5) or (6-16). Also, in the plasmon-polariton waveguide we have only TM modes (Fig.1 (a)-(c)) where the electric field is perpendicular to the interface layers, but for the last step we use a behaviour as for TE modes due to parallel orientation of the electric field with the interfaces.

For a box shaped waveguide we use $N = 3$ (Fig. 3 b and Fig. 3 d) and $N = 2$ (Fig. 3 c) and we have TE and TM modes.

3. Variational method for optical rib and box shaped waveguides

The first step in the variational method is to compute a lateral variational effective index β_l/k based on the reference central planar waveguide with the reference propagation constants β_r. In new functional we use the same reference wavefunctions from the reference region but with the original refractive index. Thus, for rib waveguide(Fig. 4) we have

$$I_\ell = \int_{-\infty}^{d_1} \frac{f_s^2}{n_s^{2\xi}} dx + \int_{d_1}^{d_2} \frac{f_1^2}{n_1^{2\xi}} dx + \int_{d_2}^{d_3} \frac{f_1^2}{n_1^{2\xi}} dx + \int_{d_3}^{\infty} \frac{f_c^2}{n_c^{2\xi}} dx, \tag{17}$$

$$J_\ell = \int_{-\infty}^{d_1} [-\frac{f_s'^2}{n_s^{2\xi}} + (kn_s^{1-\xi} f_s)^2] dx + \int_{d_1}^{d_2} [-\frac{f_1'^2}{n_1^{2\xi}} + (kn_1^{1-\xi} f_1)^2] dx + \int_{d_2}^{d_3} [-\frac{f_1'^2}{n_c^{2\xi}} + (kn_c^{1-\xi} f_1)^2] dx$$
$$+ \int_{d_3}^{\infty} [-\frac{f_c'^2}{n_c^{2\xi}} + (kn_c^{1-\xi} f_c)^2] dx , \tag{18}$$

and the variational lateral effective index β_l/k is given by

$$\frac{\beta_l}{k}(\text{var}) = \frac{\sqrt{\frac{J_\ell}{I_\ell}}}{k} , \tag{19}$$

where J_ℓ and I_ℓ are calculated with β_r.

For a dielectric loaded surface plasmon polariton rib waveguide (Fig. 5) we have

$$I_\ell = \int_{-\infty}^{d_1} \frac{f_s^2}{n_s^{2\xi}} dx + \int_{d_1}^{d_2} \frac{f_1^2}{n_1^{2\xi}} dx + \int_{d_2}^{d_3} \frac{f_2^2}{n_2^{2\xi}} dx + \int_{d_3}^{\infty} \frac{f_c^2}{n_c^{2\xi}} dx, \tag{20}$$

$$J_\ell = \int_{-\infty}^{d_1} [-\frac{f_s'^2}{n_s^{2\xi}} + (kn_s^{1-\xi} f_s)^2] dx + \int_{d_1}^{d_2} [-\frac{f_1'^2}{n_1^{2\xi}} + (kn_1^{1-\xi} f_1)^2] dx + \int_{d_2}^{d_3} [-\frac{f_2'^2}{n_c^{2\xi}} + (kn_c^{1-\xi} f_2)^2] dx \tag{21}$$
$$+ \int_{d_3}^{\infty} [-\frac{f_c'^2}{n_c^{2\xi}} + (kn_c^{1-\xi} f_c)^2] dx ,$$

and the variational lateral effective index β_l/k is given by

$$\frac{\beta_l}{k}(\text{var}) = \frac{\sqrt{\frac{J_\ell}{I_\ell}}}{k} , \tag{22}$$

where J_ℓ and I_ℓ are calculated with β_r.

For a box shaped waveguide we have to compute two lateral variational effective index β_l/k and β_{ll}/k based on the reference central planar waveguide with the reference propagation constants β_r

$$I_\ell = \int_{-\infty}^{d_1} \frac{f_s^2}{n_s^{2\xi}} dx + \int_{d_1}^{d_2} \frac{f_1^2}{n_1^{2\xi}} dx + \int_{d_2}^{d_3} \frac{f_2^2}{n_2^{2\xi}} dx + \int_{d_3}^{d_4} \frac{f_3^2}{n_3^{2\xi}} dx + \int_{d_4}^{\infty} \frac{f_c^2}{n_c^{2\xi}} dx, \tag{23}$$

$$J_{\ell}=\int_{-\infty}^{d_1}\left[-\frac{f_s'^{\,2}}{n_s^{2\xi}}+(kn_s^{1-\xi}f_s)^2\right]dx+\int_{d_1}^{d_2}\left[-\frac{f_1'^{\,2}}{n_1^{2\xi}}+(kn_1^{1-\xi}f_1)^2\right]dx+\int_{d_2}^{d_3}\left[-\frac{f_2'^{\,2}}{n_1^{2\xi}}+(kn_1^{1-\xi}f_2)^2\right]dx$$
$$+\int_{d_3}^{d_4}\left[-\frac{f_3'^{\,2}}{n_1^{2\xi}}+(kn_1^{1-\xi}f_3)^2\right]dx+\int_{d_4}^{\infty}\left[-\frac{f_c'^{\,2}}{n_c^{2\xi}}+(kn_c^{1-\xi}f_c)^2\right]dx, \tag{24}$$

$$I_{\ell\ell}=\int_{-\infty}^{d_1}\frac{f_s^2}{n_s^{2\xi}}dx+\int_{d_1}^{d_2}\frac{f_1^2}{n_1^{2\xi}}dx+\int_{d_2}^{d_3}\frac{f_2^2}{n_2^{2\xi}}dx+\int_{d_3}^{d_4}\frac{f_3^2}{n_3^{2\xi}}dx+\int_{d_4}^{\infty}\frac{f_c^2}{n_c^{2\xi}}dx, \tag{25}$$

$$J_{\ell\ell}=\int_{-\infty}^{d_1}\left[-\frac{f_s'^{\,2}}{n_s^{2\xi}}+(kn_s^{1-\xi}f_s)^2\right]dx+\int_{d_1}^{d_2}\left[-\frac{f_1'^{\,2}}{n_s^{2\xi}}+(kn_s^{1-\xi}f_1)^2\right]dx+\int_{d_2}^{d_3}\left[-\frac{f_2'^{\,2}}{n_s^{2\xi}}+(kn_s^{1-\xi}f_2)^2\right]dx$$
$$+\int_{d_3}^{d_4}\left[-\frac{f_3'^{\,2}}{n_s^{2\xi}}+(kn_s^{1-\xi}f_3)^2\right]dx+\int_{d_4}^{\infty}\left[-\frac{f_c'^{\,2}}{n_c^{2\xi}}+(kn_c^{1-\xi}f_c)^2\right]dx, \tag{26}$$

and the variational lateral effective index β_l/k and β_{ll}/k are given by

$$\frac{\beta_l}{k}(\text{var})=\frac{\sqrt{\frac{J_\ell}{I_\ell}}}{k}, \tag{27}$$

$$\frac{\beta_{ll}}{k}(\text{var})=\frac{\sqrt{\frac{J_{\ell\ell}}{I_{\ell\ell}}}}{k}, \tag{28}$$

where J_ℓ and I_ℓ, $J_{\ell\ell}$ and $I_{\ell\ell}$ are calculated with β_r.

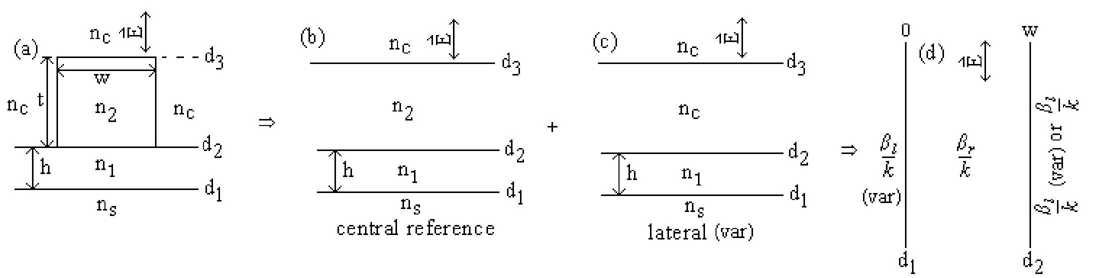

Fig. 5. Cross section of a dielectric loaded surface plasmon polariton rib waveguide and slab waveguide variational components.

The next step is to compute the propagation constant β of the rib or dielectric loaded surface plasmon polariton rib waveguides by using the new symmetric slab waveguide with three layers with the variational effective index β_l/k (var), β_r/k, and β_l/k (var), (Figs. 4-5 (d)). For a box shaped waveguide we use a new symmetric slab waveguide with five layers with two lateral variational effective index β_l/k (var) and β_{ll}/k (var) and the reference central index effective β_r/k (Fig. 6 (d)).

4. Combination of the variational and effective index methods for optical rib and box shaped waveguides

In the combinated method we use in the final step, an asymmetric slab waveguide with the following sequence of the index effective β_l/k (var), β_r/k, β_l/k for the rib or dielectric loaded surface plasmon polariton rib waveguides and two variants for a box shaped waveguide: β_{ll}/k (var), β_l/k (var), β_r/k, β_l/k (var), n (SiO_2) abbreviated as VM-EIM$_1$ method or β_{ll}/k (var), β_l/k (var), β_r/k, β_l/k, n (SiO_2) abbreviated as VM-EIM$_2$ method.

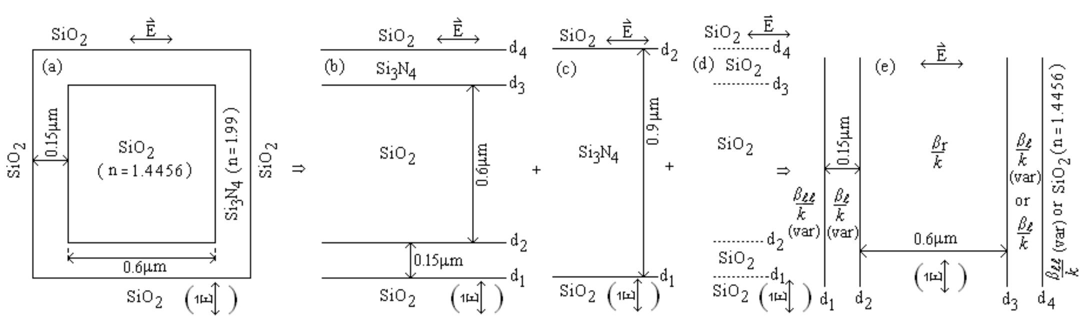

Fig. 6. Cross section of a box shaped waveguide and slab waveguide variational components.

5. Numerical results and conclusions

The usual effective index (EIM), the variational effective index (VM) and combinated (VM-EIM) methods has been applied to two test optical rib waveguides and to a box shaped waveguide.

For a optical rib waveguide (Fig.1 and Fig.4) with a fixed rib width w = 3μm and for different values of the rib high h, the computed values of the normalized propagation constant $b = (\beta^2 / k^2 - n_S{}^2)/(n_1{}^2 - n_S{}^2)$ are shown in Table 1 and are compared with results obtained using a recently optimized variational iterative method [1] and with the finite element method (FEM). It can be seen that our combinated method gives normalized propagation constants which are closer to the benchmark finite element method.

Table 1. Variation of the normalized propagation constant b of the fundamental mode TE_0 with optical rib high h for a wavelength λ = 1.15μm and n_s = 3.4, n_1 = 3.44, n_c = 1, h + t = 1μm.

h(μm)	EIM	VM	IV_{opt}[1]	VM-EIM	FEM[1]
0.5	0.3556	0.3109	0.3183	0.3375	0.3293
0.6	0.3586	0.3154	0.3255	0.3412	0.3394
0.7	0.3652	0.3228	0.3335	0.3484	0.3518
0.8	0.3571	0.3364	0.3473	0.3603	0.3669
0.9	0.3909	0.3648	0.3750	0.3812	0.3872

Table 2. Variation of the real part of the effective indexes of the fundamental mode TM_0 with the thickness t and width w of the dielectric loaded surface plasmon polariton rib for a wavelength λ = 1.55μm and n_s = 1.6, n_1 = 0.55 + 11.5i, n_2 = 1.535 n_c = 1.

t(μm)×w(μm)	EIM	VM	VM-EIM	FEM[2]
0.6×0.6	1.3054	1.2898	1.2979	1.291
0.6×0.5	1.2685	1.2467	1.2582	1.250
0.6×0.4	1.2213	1.1900	1.2067	1.198
0.6×0.3	1.1622	1.1160	1.1413	1.133
0.6×0.2	1.0940	1.0253	1.0653	1.064
0.5×0.5	1.2442	1.2143	1.2303	1.221
0.4×0.5	1.2045	1.1614	1.1853	1.176
0.3×0.5	1.1402	1.0759	1.1138	1.111
0.2×0.5	1.0566	0.9754	1.0295	1.041

For a dielectric loaded surface plasmon polariton rib waveguide (Fig.2 and Fig.5) with the thickness t and width w, for a fixed value of rib high h = 0.1μm, the computed values of the mode effective indexes and propagation lengths $1/(2\beta'')$ of the single fundamental mode TM_0 are shown in Table 2 and Table 3, respectively and are compared with results obtained using a recently finite element method (FEM) where β'' is the imaginary part of the complex propagation constant $\beta = \beta' - i\beta''$. The mode effective indexes calculated by means of the EIM are larger than those calculated by means of the VM, VM-EIM and FEM. The mode effective indexes and propagation lengths calculated by means of the VM are smaller than those calculated by means of the EIM, VM-EIM and FEM. It can be seen that our combinated method gives mode effective indexes and propagation lengths which are closer to the finite element method.

Table 3. Variation of the propagation lengths of the fundamental mode TM_0 with the thickness t and width w of the dielectric loaded surface plasmon polariton rib for a wavelength λ = 1.55μm and n_s = 1.6, n_1 = 0.55 + 11.5i, n_2 = 1.535 n_c = 1.

t(μm)×w(μm)	EIM	VM	VM-EIM	FEM[2]
0.6×0.6	45.2323	42.5948	43.9941	44.4
0.6×0.5	47.8033	43.9998	46.0443	46.4
0.6×0.4	52.6644	46.7487	50.0029	50.1
0.6×0.3	62.9939	52.6056	58.6177	59.1
0.6×0.2	89.2891	66.7033	81.9810	81.3
0.5×0.5	43.4689	39.3008	41.6288	42.2
0.4×0.5	39.8691	35.2651	38.0439	39.5
0.3×0.5	39.7831	34.7721	38.5187	41.7
0.2×0.5	59.1569	63.3985	67.1027	66.0

Also, our combined method is applied to determine the effective index of a box shaped waveguide (Fig. 3 and Fig. 6) where a thin high-index coating boxing a low – index inner material [3]. In this model, the Si_3N_4 (n = 1.99) shell around of h ×h μm^2 of SiO_2 (n = 1.4456) core is surrounded by SiO_2. It can be seen that our combinated method gives mode effective indexes which are closer to the coupled mode theory [3], Beam Propagation Method (BPM) [4], Film Mode Matching Method (FMM)[5] and Variational Effective Index Method (VEIM) [5]. In the FMM and VEIM methods, the results are obtained with a large number (30) of TE/TM modes in expansion.

Table 4. Variation of the effective indexes of the fundamental TE_0 and TM_0 modes with the thickness t of the nitride layer and inner width h of the low core index for a box shaped waveguide for a wavelength λ = 1.55μm.

h(μm)×t(μm)	EIM	VM	VM-EIM_1	VM-EIM_2	
0.6×0.15 TE	1.552	1.518	1.528	1.538	1.527[3] 1.521[4] 1.524[5]
0.6×0.15 TM	1.546	1.497	-	1.523	1.527[3] 1.521[4] 1.524[5]
1.1×0.065 TE	1.472	1.459	1.463	1.466	1.465[4]
1.1×0.065TM	1.475	1.457	-	1.466	1.465[4]

We can easily generate another "mid-way" value by simply taking the average of the solutions from the effective index method and the variational method. Here we should point out that a "mid-way" value is not necessarily more accurate, because the two methods have different degrees of accuracy. This can be seen, for example, from Table 1 for the rib waveguide with h=0.9um, where the effective index solution (EIM) is actually more accurate than any of the other listed approximations. We should also mention that the effective index method does not always overestimate the propagation constant. The accuracy of the method depends critically on the waveguide geometry and the operation conditions.

Our simple method combines the advantage of the effective index and variational methods and is important for engineering project of multilayer waveguides with 2-D index profiles.

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