

# Flexural vibrations band gaps in phononic crystal Timoshenko beam by plane wave expansion method

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With the idea of the phononic crystals, the advanced composite structures of phononic crystal Timoshenko beams are designed. The simplified vibration equation and the general vibration equation are firstly presented in this paper. Based on the two vibration equations, the plane wave expansion method is introduced to calculate the bending vibration band structures of phononic crystal Timoshenko beams. Compared with the results of two methods and the exact solution by the transfer matrix method, the present plane wave expansion method with two variables is suitable for solving band gaps of phononic crystal Timoshenko beams.

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## 1. Introduction

In the recent years, the existence of complete band gaps in periodic composite materials called phononic crystal (PC) has received much attention. According to its adjustable frequency ranges of band gaps, the PC systems have widely potential applications in controlling vibrations. Based on the relevant scientific research, some PCs have been successfully designed as high efficiency waveguides [1, 2] and transducers [3, 4], while more potential engineering applications of the PCs, e.g., vibration isolation, acoustic collimating and noise reduction will be apperceived.

The emphasis of band gaps in PCs is laid on both experimental and theoretical research. A large count of tests with different periodic structures from 1D to 3D has been performed, and two kinds of gap mechanism for PCs, Bragg scattering mechanism [5, 6] and locally resonant mechanism [7, 8] have been observed. Meanwhile, several theoretical methods have been developed to study the band structures, such as the transfer matrix method (TMM) [9], the plane wave expansion method (PWEM) [5, 6, 10], the finite difference time domain method (FDTD) [1, 11, 12], the multiple scattering theory (MST) [13, 14], the lumped-mass method (LMT) [15], and the finite element method (FEM) [16].

Beams are often applied in bridges, railways, and the part of sustaining constructs while the dynamic analysis of beams has been a pop topic [17], in which the problem of resonance vibration has been often encountered. Due to the existence of band gaps in PCs, the concept of PC beams provides a possible approach for the control of

vibration. Wen et al. [18] calculated the flexural vibration band structure of PC Euler-Bernoulli beam by PWEM. Yu et al. [19] presented the flexural vibration band structure of PC Timoshenko beam by PWEM with single variable and compared with the results of a finite periodic structure by FEM. By using TMM, Shen et al. [20] studied the flexural vibration property of periodic pipe beam conveying fluid based on Timoshenko beam and the effects of the spring support stiffness on the band gap have been analyzed. Xiang et al. [21] calculated the flexural vibration band structure of PC Timoshenko/Euler-Bernoulli beams by differential quadrature element method (DQEM).

Having the distinct physical meanings, PWEM is the most popular method to calculate the band structures of PCs. Existing the faire calculating error (maximum relative error is about 15% in the first 3 band gaps) by PWEM with single variable in the relevant research, the flexural vibration band structure of PC Timoshenko beam is analyzed in this paper. Firstly, the simplified vibration equation is presented according to the assumption of the homogenization. Based on the basic assumption of Timoshenko beam theory, the general governing equation of flexural vibration of heterogeneous Timoshenko beam with two variables is deduced. By introducing the idea of Fourier series and Bloch theorem, PWEM with two variables or one, referred to the general vibration equation or simplified equation, is presented to solve the vibration equations. The flexural vibration band structures of PC Timoshenko beams are calibrated by solving the algebraic eigenvalue equations. The comparisons between the different methods are performed finally.

## 2. Vibration equation

### 2.1 Simplified vibration equation

In many conditions, homogenous beams are applied in many engineering applications, and periodic structures are regarded as homogenous systems at local level, so the general vibration equation can be deduced as a simplified formation. To solve the flexural vibration band gaps of PC Timoshenko beams, the general vibration equation for TMM can be developed by the vibration equation of a homogenous Timoshenko beam:

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w(x,t)}{\partial x^2} \right) + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} - \rho I \left( 1 + \frac{E}{\kappa G} \right) \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G} \frac{\partial^4 w(x,t)}{\partial t^2} = 0 \quad (1)$$

where  $E$  and  $G$  represent the Young’s modulus and shear modulus of beam respectively.  $I$  is the second axial moment of area, and  $A$  is the area of cross section of the beam.  $w(x, t)$  is the flexural displacement. Although a PC Timoshenko beams describes the heterogeneous behavior, the physical parameters  $E(x)$ ,  $G(x)$ ,  $I(x)$  and  $A(x)$  can be regarded as invariables in Eq. (1). The reason is that Eq. (1) can be regarded as the control equation for each segment and TMM is applied at the interface of the segment directly. That is to say the material parameters in Eq. (1) keep constant in the range of the segment. Evidently, the above vibration equation has clear physical meanings for TMM which has been used to calculate the band structures of PC Timoshenko beams. The result of TMM is the exact solution and it is considered as analytical solution.

Assuming that a PC Timoshenko beam is regarded as a homogenous system at macroscopic level, the simplified vibration equation for PWEM can be expressed as following:

$$\frac{\partial^2}{\partial x^2} \left( E(x)I(x) \frac{\partial^2 w(x,t)}{\partial x^2} \right) + \rho(x)A(x) \frac{\partial^2 w(x,t)}{\partial t^2} - \rho(x)I(x) \left( 1 + \frac{E(x)}{\kappa(x)G(x)} \right) \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + \frac{\rho(x)^2 I(x)}{\kappa(x)G(x)} \frac{\partial^4 w(x,t)}{\partial t^2} = 0 \quad (2)$$

This simplified vibration equation for PWEM, described by the flexural displacement  $w(x, t)$ , has been proposed to solve the PC Timoshenko beam [19].

### 2.2 General vibration equation

Timoshenko beam theory [17] takes the effects of shear deformation and rotational inertia into account, and the flexural vibration of a PC Timoshenko beam should be regarded as a problem of a heterogeneous Timoshenko beam. Therefore, the material parameters and physical variables are related to the coordinate  $x$ . According to

Timoshenko beam theory, the cross section of beam keeps being a plane after deformation. Therefore, the angle caused by shear deformation  $\beta(x, t)$  can be expressed as:

$$\beta(x,t) = \frac{\partial w(x,t)}{\partial x} - \alpha(x,t) \quad (3)$$

where  $\alpha(x, t)$  is the slope of neutral axis.

Considering the different location, the bending moment  $M(x, t)$  and the shear force  $Q(x, t)$  of cross section can be expressed as:

$$\begin{cases} M(x,t) = -E(x)I(x) \frac{\partial \alpha(x,t)}{\partial x} \\ Q(x,t) = \kappa(x)A(x)G(x) \left( \frac{\partial w(x,t)}{\partial x} - \alpha(x,t) \right) \end{cases} \quad (4)$$

Note that  $E(x)$ ,  $G(x)$ ,  $I(x)$  and  $A(x)$  should be expressed by the function of  $x$ .  $\kappa(x)$  is shear correction coefficient which is relevant to the shape of cross section. It is equal to 5/6 in the case of rectangle cross section.

The conditions for rotational equilibrium and the force equilibrium require that:

$$\begin{cases} \frac{\partial M(x,t)}{\partial x} - Q(x,t) + m(x)r(x)^2 \frac{\partial^2 \alpha(x,t)}{\partial t^2} = 0 \\ \frac{\partial Q(x,t)}{\partial x} - m(x) \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \end{cases} \quad (5)$$

where  $m(x)=\rho(x)A(x)$  is the mass per unit length,  $r(x)=(I(x)/A(x))^{1/2}$  is radius of gyration of cross section. By substituting Eq. (4) into Eq. (5), the general vibration equation of a heterogeneous Timoshenko beam can be expressed as following:

$$\begin{cases} \frac{\partial}{\partial x} \left( E(x)I(x) \frac{\partial \alpha(x,t)}{\partial x} \right) + \kappa(x)A(x)G(x) \left( \frac{\partial w(x,t)}{\partial x} - \alpha(x,t) \right) - \rho(x)I(x) \frac{\partial^2 \alpha(x,t)}{\partial t^2} = 0 \\ \kappa(x)A(x)G(x) \left( \frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial \alpha(x,t)}{\partial x} \right) - \rho(x)A(x) \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \end{cases} \quad (6)$$

It is evident that to solve the Eq. (6), two coupled variables, flexural displacement  $w(x, t)$  and rotational angle  $\alpha(x, t)$ , should be taken into account simultaneously. By using the general vibration equation, the case of PC Timoshenko beams can be well calculated.

## 3. Plane wave expansion method

Fig. 1 illustrates a binary PC Timoshenko beam. The

geometric parameters (cross section and length) and material parameters (Young's modulus, shear modulus and density) of segmental beams are periodically arranged along  $x$  direction infinitely. In a period, there are two segmental beams. The geometric and material parameters are constant in each segmental beam.  $a_1$  and  $a_2$  are the length of segmental beam A and B, respectively.  $a=a_1+a_2$  is the periodic parameter of binary PC Timoshenko beam.

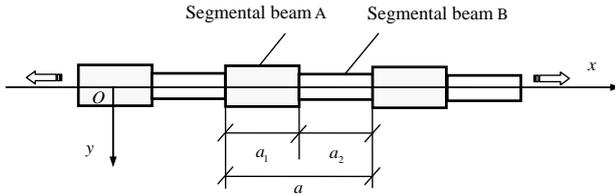


Fig. 1. Binary PC Timoshenko beam.

### 3.1 General method with two variables

For the general vibration equation Eq. (6) of PC Timoshenko beam, we can demonstrate  $m_1(x)=\rho(x)A(x)$ ,  $m_2(x)=\kappa(x)G(x)A(x)$ ,  $m_3(x)=\rho(x)I(x)$ ,  $m_4(x)=E(x)I(x)$ . Because of the periodicity of PC Timoshenko beam, these material parameters can be expanded in Fourier series as:

$$m_n(x) = \sum_{G_2} M_n(G_2) \exp(iG_2 x) \quad n=1,2,3,4 \quad (7)$$

where  $G_2$  is one-dimensional reciprocal lattice vector and  $M_n(G_2)$  is the corresponding Fourier coefficient of material parameters  $m_n(x)$ .

By means of Bloch theorem, the flexural displacement  $w$  and rotational angle of cross section  $\alpha$  can be expressed as:

$$\begin{cases} w(x,t) = \exp[i(kx - \omega t)] w_k(x) \\ \alpha(x,t) = \exp[i(kx - \omega t)] \alpha_k(x) \end{cases} \quad (8)$$

where  $k$  is one-dimensional wave vector and  $\omega$  is circular frequency.  $w_k$  and  $\alpha_k$  are periodic functions with the same spatial periodicity of material parameters, so  $w_k$  and  $\alpha_k$  also can be expanded in Fourier series:

$$\begin{cases} w_k(x) = \sum_{G_1} W_k(G_1) \exp(iG_1 x) \\ \alpha_k(x) = \sum_{G_1} \theta_k(G_1) \exp(iG_1 x) \end{cases} \quad (9)$$

where  $G_1$  is also one-dimensional reciprocal lattice vector.  $W_k(G_1)$  and  $\theta_k(G_1)$  are Fourier coefficients of  $w_k$  and  $\alpha_k$ , respectively.

By substituting Eqs. (7), (8) and (9) into Eq. (6), we obtain:

$$\begin{cases} \omega^2 \sum_{G_1} M_1(G_3 - G_1) W_k(G_1) = \sum_{G_1} (k + G_1)(k + G_3) \\ \quad M_2(G_3 - G_1) W_k(G_1) + \\ \quad \sum_{G_1} [i(k + G_3)] M_2(G_3 - G_1) \theta_k(G_1) \\ \omega^2 \sum_{G_1} M_3(G_3 - G_1) \theta_k(G_1) = \\ \sum_{G_1} [(k + G_1)(k + G_3) M_4(G_3 - G_1) + M_2(G_3 - G_1)] \\ \quad \theta_k(G_1) - \sum_{G_1} [i(k + G_1)] M_2(G_3 - G_1) W_k(G_1) \end{cases} \quad (10)$$

where  $G_3 = G_1 + G_2$ . Equation (10) is an infinite-order eigen-value problem, so the Fourier series need to be translated to finite items to obtain the numerical solution. Generally, the result is more precise with the greater value of  $n$ . When  $n$  reciprocal vectors are selected in primitive reciprocal lattice vector direction,  $2n+1$  plane waves are chosen. A  $2 \times (2n+1)$  matrix equation can be expressed as a generalized eigenvalue equation:

$$\omega^2 \mathbf{P} \mathbf{X} = \mathbf{Q} \mathbf{X} \quad (11)$$

where  $\mathbf{X} = [W_k(G_1) \ \theta_k(G_1)]^T$ ,  $\mathbf{P} = [P^{ij}]$  and  $\mathbf{Q} = [Q^{ij}]$  with  $i, j=1,2$ .  $W_k(G_1)$  and  $\theta_k(G_1)$  are both  $2n+1$  order arrays of Fourier coefficients.  $P^{ij}$  and  $Q^{ij}$  are all  $2n+1$  order square block matrices and the entries of them are:

$$\begin{aligned} P_{ij}^{11} &= M_1(G_3^{(i)} - G_1^{(j)}) \\ P_{ij}^{12} &= P_{ij}^{21} = 0 \\ P_{ij}^{22} &= M_3(G_3^{(i)} - G_1^{(j)}) \\ Q_{ij}^{11} &= (k + G_1^{(j)})(k + G_3^{(i)}) M_2(G_3^{(i)} - G_1^{(j)}) \\ Q_{ij}^{12} &= i(k + G_3^{(i)}) M_2(G_3^{(i)} - G_1^{(j)}) \\ Q_{ij}^{21} &= -i(k + G_1^{(i)}) M_2(G_3^{(i)} - G_1^{(j)}) \\ Q_{ij}^{22} &= (k + G_1^{(j)})(k + G_3^{(i)}) M_4(G_3^{(i)} - G_1^{(j)}) + \\ &\quad M_2(G_3^{(i)} - G_1^{(j)}) \end{aligned} \quad (12)$$

where  $G_1^{(j)}$  is the reciprocal vector in the  $G_1$  reciprocal vector space with index  $j$ , and  $G_3^{(i)}$  is the reciprocal vector in the  $G_3$  reciprocal vector space with index  $i$ . Because  $\mathbf{P}$  is a real symmetric positive definite matrix and  $\mathbf{Q}$  is a Hermitian matrix, eigenvalue  $\omega^2$  is real based on matrix theory. If wave vector  $k$  is picked from the first Brillouin zone which is interval  $[-\pi/a, \pi/a]$ , Equation (11) can be solved numerically. Therefore, the relation curves between wave vector  $k$  and eigen-frequency  $\omega$  are obtained,  $\square$  which is the band structure of PC Timoshenko beam.

### 3.2 Simplified method with one variable

Based on the simplified vibration equation, the similar method is performed and the generalized eigenvalue equation can be deduced by using the similar method calculated for the general vibration equation of PC Timoshenko beam. This method has been firstly proposed by Yu et al. In this paper, we just propose this simplified method directly. More details can be seen in Ref [19].

### 4. Numerical calibration

To validate the proposed methods for calculating the band structures of PC Timoshenko beams, one example is calculated. Material A and B are aluminum and epoxy. The elastic parameters employed in the calculations are  $\rho_A=2730 \text{ kg/m}^3$ ,  $E_A=77.56 \text{ GPa}$ ,  $G_A=28.87 \text{ GPa}$ ,  $\rho_B=1180 \text{ kg/m}^3$ ,  $E_B=4.35 \text{ GPa}$  and  $G_B=1.59 \text{ GPa}$ . The geometric parameters employed in the calculations are  $l_A=l_B=0.035 \text{ m}$ ,  $b_A=b_B=0.02 \text{ m}$  and  $h_A=h_B=0.01 \text{ m}$ . The above mentioned parameters were also used by Yu et al. [19] that helps to verify the results of the proposed methods.

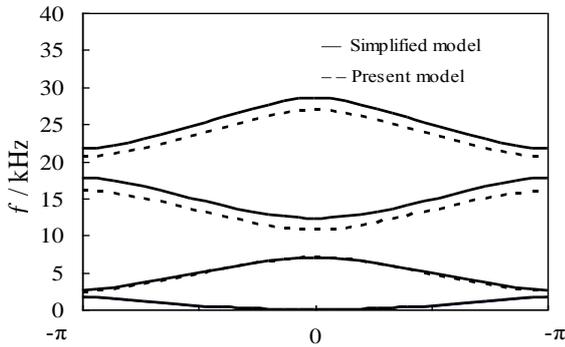


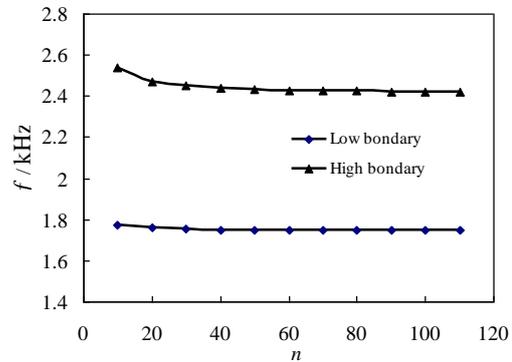
Fig. 2. Band structure of a PC Timoshenko beam.

Table 1. Boundaries and the relative errors of the first 3 band gaps of the PC Timoshenko beam by different methods.

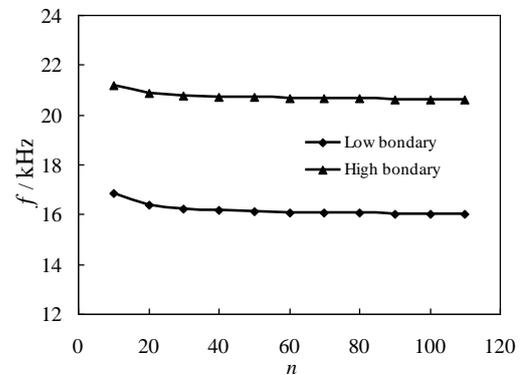
		Simplified method[18]		Present method		TMM
		Hz	%	Hz	%	
BG 1	Low boundary	1684	4.0	1747	0.2	1744
	High boundary	2610	8.5	2421	0.6	2406
BG 2	Low boundary	7014	0.2	7063	0.5	7030
	High boundary	12300	15.4	10712	0.5	10658
BG 3	Low boundary	17750	11.4	16025	0.6	15934
	High boundary	21642	5.3	20622	0.3	20554

Fig. 2 shows the band structures with  $n=100$ . Continuous line is the result by using the simplified

vibration equation and dashed line is the result of the general vibration equation. To well describe the influence of the two methods, Table 1 shows the boundaries and the relative errors of the first 3 band gaps. Being an analytical method, the result of TMM is more accurate than that of the other methods, and its value is regarded as the standard solution. We can see that for the first 2 band gaps, the result of the simplified method has larger range of frequency while its low boundary is lower and its high boundary is higher than that of the general method. The trend of the third band gap shows that the low boundary and the high boundaries of the simplified method are superior to that of the general method. Furthermore, compared with the results of TMM, the maximum relative error of the general method is about 0.6% while its value of the simplified method is about 15%. It is considered that the stability of the general method is better than that of the simplified method. The reason is that the simplified vibration equation is deduced by the assumption of homogenization; however, the material parameters of composites are significant different in this sample, which leads to inhomogeneous behavior at local level. It is concluded that the simplified method has fair relative error in some condition where the PC has large contrast in physical properties between the composites.



(a) First band gap



(b) Third band gap

Fig. 3. The trend of eigen-frequencies calculated by different numbers of plane waves.

Table 2. The computational time and the maximum relative error of BG 3 with different numbers of plane waves.

$n$	10	20	30	40	50	60
Time (s)	0.12	0.72	2.21	4.88	9.15	16.32
Relative errors (%)	3.10	2.86	1.90	1.42	1.14	0.95
$n$	70	80	90	100	110	
Time (s)	28.70	42.58	61.70	87.94	115.47	
Relative errors (%)	0.81	0.71	0.63	0.57	0.52	

Then the different numbers of plane wave are selected as  $n=10, 20 \dots 110$  and we calculate the band structure and the maximum relative error by the general vibration equation. Fig. 3 shows the boundaries of the first and the third band gap with the different  $n$  while Table 2 presents the computational time and the maximum relative error of the BG 3 with the different  $n$ . It is concluded as following:

*a.* The PWE method converges downward to the exact solution as the number of plane wave increases. When the number of plane waves reaches 60, the relative error can be kept less than 1.0%.

*b.* Results show that when the value of  $n$  is very large, the speed of convergence will be low due to the well-known Gibbs oscillations at the interfaces. At the same time, it takes a mount of time to calculate because the order of matrix of Eq. (11) is  $4n+2$ , that is to say when  $n$  increases, the matrix calculation processes will increase by power series.

Considering the relative error and computational time, a set of 100 plane waves is considered to employ in the calculations.

## 5. Conclusion

PWEM with two coupled variables (bending displacement and rotational angle) in calculation the bending vibration band structure of PC Timoshenko beam is proposed. Examples are studied to show its correctness and advantages in comparison with the PWEM with one variable (bending displacement). The present method shows better computational accuracy. Due to the advantages of the distinct physical meanings of PWEM with two coupled variables, the present method is easy to be extended to other situations of PC Timoshenko beam with few modifications.

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