Conservation laws for cubic–quartic optical solitons with Fokas–Lenells equation having maximum intensity

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This paper retrieves conservation laws of the cubic–quartic Fokas–Lenells equation with power–law of nonlinear refractive index. The multiplier approach reveals three conservation laws. The conserved quantities are computed from the bright 1–soliton solutions that are reported earlier.

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1. Introduction

One of the essential features of any optoelectronic system is its conservation laws. Once these laws are identified several additional features of the system naturally emerge. These include the analytical study of four–wave mixing (4WM), collision–induced timing jitter, analysis of the Langevin equation for additive stochastic perturbation and many others. Therefore, it is imperative to address the conservation laws of any optoelectronic system. Fokas–lenells equation (FLE) in optical fibers is one of the latest models that has been studied to address dispersive optical solitons [1–15].

The current paper therefore discusses and derives the conserved densities and hence the conserved quantities of cubic–quartic (CQ) dispersive optical solitons that are modeled by FLE with power–law of nonlinear refractive index. The concept of CQ solitons is fairly new as well. These emerge when chromatic dispersion is replaced by the combination of third–order and fourth–order dispersions and hence the name. CQ–FLE has been studied during recent times and its soliton solutions have been successfully recovered [1, 4, 5]. However, this model with power–law nonlinearity has been just introduced and its bright soliton solutions have also been recently reported [4]. The current paper moves a small step ahead. The conservation laws for CQ–FLE with power–law form of self–phase modulation (SPM) are derived and the corresponding conserved quantities are exhibited. The multiplier approach brings about success in the conservation laws retrieval.

1.1. Governing model

The cubic–quartic perturbed Fokas–Lenells equation with power law nonlinearity is written as

$$iq_t + ia_3 q_{xxx} + a_4 q_{xxxx} + |q|^{2m}(bq + icq_x) = i[\lambda(|q|^{2m}q)_x + \mu(|q|^{2m})_x q], \tag{1}$$

where a_3, a_4, b, c, λ and μ are real–valued constants, while $q(x,t)$ is a complex–valued function that represents the wave profile. x and t are the non–dimensional spatial and temporal variables, respectively. The first term in Eq. (1) corresponds to linear temporal evolution, where $i = \sqrt{-1}$. Then, a_3 and a_4 are the coefficients of third-order dispersion and fourth-order dispersion, in sequence. Next, b, c, λ and μ imply respectively the coefficients of power–law nonlinearity, nonlinear dispersion, self–steepening for short pulses and self–frequency shift. Finally, m is the power–law nonlinearity exponent that represents maximum allowable intensity so that the laser does not burn down.

The bright 1–soliton solution to the CQ–FLE with power–law nonlinearity given by (1) is:

$$q(x,t) = A\operatorname{sech}^{\frac{2}{m}}[B(x - vt)]e^{i(-\kappa x - \omega t + \theta_0)}, \tag{2}$$

where

$$v = 4a_4\kappa^3 - 3a_3\kappa^2. \tag{3}$$

Here, A is the soliton amplitude, B is its inverse width and v is the speed at which the soliton travels. From the phase component, ω is the wave number of the soliton, while κ is the soliton frequency and θ_0 is the phase constant.

2. Conservation laws

For the conserved flow that renders a closed form of the respective partial differential equations (PDEs), we let $q = u + iv$ and split the PDEs into a real system whose conserved vectors, (T^t, T^x) satisfies the $D_t T^t + D_x T^x = 0$ along the solutions of the PDEs. It turns out that all systems lead to a conserved **power density**, viz.,

$$\Phi_P^t = \frac{1}{2}|q|^2. \tag{4}$$

Next, if

$$\lambda = -\mu, \tag{5}$$

the **linear momentum density** is given by

$$\Phi_M^t = I(q^* q_x). \tag{6}$$

Finally, the **Hamiltonian densities** are given by

$$\Phi_H^t = -\frac{1}{2}\left[\begin{array}{c}\frac{1}{(m+2)}|q|^m\{bR(qq_t^*) + cI(q_t^* q_x)\} \\ +a_3 I(q^* q_{xxx}) - a_4 R(qq_{xxxx}^*)\end{array}\right]$$

$$-\frac{1}{2(m+1)}\mu|q|^m I(q^* q_x). \tag{7}$$

The conserved quantities are now given as:

$$P = \int_{-\infty}^{\infty} \Phi_P^t dx = \int_{-\infty}^{\infty} |q|^2 dx = \frac{A^2}{B}\frac{\Gamma\left(\frac{2}{m}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{m}+\frac{1}{2}\right)}, \tag{8}$$

$$M = \int_{-\infty}^{\infty} \Phi_M^t dx = \frac{1}{2i}\int_{-\infty}^{\infty} (q^* q_x - qq_x^*)dx$$

$$= \frac{\kappa A^2}{B}\frac{\Gamma\left(\frac{2}{m}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{m}+\frac{1}{2}\right)}, \tag{9}$$

and

$$H = \int_{-\infty}^{\infty} \Phi_H^t dx$$

$$= \left[-\frac{a_3 \kappa A^2}{m(m+4)B}\{12B^2 + m(m+4)\kappa^2\}\right.$$

$$+\frac{8a_4(2m+3)A^2B^3}{m^2(m+4)(3m+4)} + \frac{12a_4\kappa^2 A^2 B}{m(m+4)}$$

$$\left.+\frac{a_4\kappa^4 A^2}{2B} + \frac{2\mu\kappa A^{m+2}}{(m+1)(m+4)B}\right]\frac{\Gamma\left(\frac{2}{m}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{m}+\frac{1}{2}\right)}. \tag{10}$$

3. Conclusions

This paper reports conserved quantities for CQ–FLE that is addressed with power–law form of SPM. The multiplier approach gives three such laws and they are the power, linear momentum and the Hamiltonian. These derived laws would next yield additional results that would lead to a new perspectives to this latest model. These would be the application of perturbation theory with quasi–monochromatic solitons to derive the adiabatic dynamics of soliton parameters that would lead to the cooling effect. Additional avenues to venture would be to study the collision–induced timing jitter, 4WM effect and several others. From the stochastic standpoint, the Langevin equation can be derived when the equation would be studied with additive noise. That would give way to the mean–free velocity of the solitons. Such studies would be aligned with the recently reported works [6-15]. These prospects are just a tip of the iceberg!

Disclosure

The authors claim there is no conflict of interest.

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