

Analogy between Lorenz-Robbins system and the laser equations

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An analogy between Lorenz-Robbins system for a convective fluid flow and Haken equations for a single mode laser is presented. The system called Lorenz-Robbins-Haken (LRH) is both analytically and numerically investigated. As the control parameter, which is proportional with the pump power, is varied, a rich behaviour of the system is observed, including the chaotic one. This behaviour corresponds to the experimental observations.

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1. Introduction

The analogy between fluid dynamics and laser instabilities is now well known. In 1963 E. Lorenz [1] obtained a system of three first order coupled nonlinear differential equations as a model for the convective flow. This system has a rich dynamics, including chaos, when the control parameter is varied. The Lorenz model has been extensively investigated analytical and numerical over the last years and there is a vast literature about it (see for ex. [2]-[5]). In 1975 it was pointed out by Haken [6], that the Lorenz equations for a convective fluid flow, and the single mode laser equations are mathematically isomorphic. This analogy between the laser equations and the Lorenz equations has been used by various scientists to investigate, for example, the route to chaos in the $^{15}\text{NH}_3$ far infrared ring laser [7], or to control chaos in laser equations [8]. These equations was called Lorenz-Haken equations [9]. But in this analogy a simplified form of the Lorenz equations was considered.

In this work we reconsider this analogy and suggest the analogy between the Lorenz-Robbins system [10], [11] and the laser equations. The system, which will be called Lorenz-Robbins-Haken (LRH), is then analysed in details, both analytically and numerical.

2. The Lorenz-Robbins-Haken equations

The laser equations are:

$$\begin{aligned} \frac{dE}{dt} &= k(P - E) \\ \frac{dP}{dt} &= \gamma_{\perp}(ED - P) \\ \frac{dD}{dt} &= \gamma_{\parallel}[(\lambda + 1) - D - \lambda EP] \end{aligned} \quad (1)$$

where E , P and D are the normalized electric field strength, the polarization and the atomic inversion. The significance of the system (1) parameters is: k - the cavity loss; γ_{\perp} - the line width; γ_{\parallel} - the inverse longitudinal

relaxation time, and $\lambda = \frac{D_0 - D_c}{D_c}$, where D_0 and D_c are

the unsaturated and critical inversion, respectively, represents the pump power.

The classical Lorenz equations are:

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= xy - bz \end{aligned} \quad (2)$$

where the variable x is proportional to the circulatory fluid flow velocity, y characterizes the temperature difference between rising and falling fluid regions, and z characterizes the distortion of the vertical temperature profile from its linear, with height, equilibrium variation. The dimensionless parameters in the equations (2) have the following connotation: $\sigma > 0$ is the Prandtl number,

$r = \frac{R}{R_c}$, with R - the Rayleigh number, is considered as

the control parameter, and $b > 0$.

Even though the Lorenz and the Haken equations are mathematically isomorphic, their conventional layout is different, as shown in the equations (1) and (2). We note that in the analogy between the laser equations and the Lorenz equations, which has been used by various scientists, a simplified form of the Lorenz equations was considered, namely:

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - z) \\ \frac{dy}{dt} &= -xz - y \\ \frac{dz}{dt} &= -r - z + xy \end{aligned} \quad (3)$$

We will consider another possible analogy between Lorenz-Robbins system [6], [7] and the laser equations. In this case both systems become identical.

Let us consider the equations of the system (2) with the following substitution of variables:

$$z = r - u; \quad x \rightarrow y; \quad y \rightarrow z, \quad (4)$$

then, with a convenient notation,

$$\begin{aligned} y &\rightarrow X \\ z &\rightarrow Y, \\ u &\rightarrow Z \end{aligned} \quad (5)$$

we obtain:

$$\begin{aligned} \frac{dX}{dt} &= \sigma(Y - X) \\ \frac{dY}{dt} &= XZ - Y \\ \frac{dZ}{dt} &= b(r - Z) - XY \end{aligned} \quad (6)$$

It can be seen that the system (1) is now identical with the system (6), by the following identification:

$$\begin{aligned} t \rightarrow t \frac{\sigma}{k}; \quad E &\rightarrow \frac{X}{\sqrt{b(r-1)}}, \quad P \rightarrow \frac{Y}{\sqrt{b(r-1)}}; \\ D &\rightarrow Z; \quad \gamma_{II} = \frac{kb}{\sigma}; \quad \gamma_{\perp} = \frac{k}{\sigma}; \quad \lambda = r - 1 \end{aligned} \quad (7)$$

The system (6) will be called Lorenz-Robbins-Haken (LRH) system.

3. The global dynamic behaviour analysis of LRH system

For simplicity, we will write the LRH system in the form:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= xz - y \\ \dot{z} &= b(r - z) - xy \end{aligned}, \quad (8)$$

where the dots represent the derivation with respect to the new time: $t \frac{\sigma}{k}$.

The system (8) is a dissipative one. Let $f = (f_1, f_2, f_3)^T = (\dot{x}, \dot{y}, \dot{z})^T$. Then from (8) we have

$$\text{div } f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -\sigma - 1 - b \quad (9)$$

If $-(\sigma + b + 1) < 0$, i.e. $\sigma + b > 0$ or

$\frac{\gamma_{\perp}}{k} + \frac{\gamma_{II}}{\gamma_{\perp}} > 0$, the LRH system contracts the space

phase volumes and could have attractors. This is our case.

The equilibrium states of the system (8), i.e. the fixed points in the system phase state, are the solutions of the equations:

$$\begin{aligned} \sigma(y - x) &= 0 \\ xz - y &= 0 \end{aligned} \quad (10)$$

$$b(r - z) - xy = 0$$

We obtain, for $\sigma \neq 0$ and $b \neq 0$:

$O_1(0, 0, r)$, which corresponds, for the laser equations, to $(0, 0, \lambda + 1)$, and also

$O_{2,3} = (\pm\sqrt{b(r-1)}; \pm\sqrt{b(r-1)}; 1)$, for $r > 1$, which corresponds, for the laser equations, to

$(\pm\sqrt{\frac{\gamma_{II}}{\gamma_{\perp}} \lambda}; \pm\sqrt{\frac{\gamma_{II}}{\gamma_{\perp}} \lambda}; 1)$ for $\lambda > 0$.

The stability of these equilibrium states is analysed by the usual technique. So, the Jacobian matrix of the system is:

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ z & -1 & x \\ -y & -x & -b \end{pmatrix} \quad \text{and}$$

$$J|_{O_1} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

Then the characteristic equation for O_1 is

$$\begin{vmatrix} -\sigma - \Lambda & \sigma & 0 \\ r & -1 - \Lambda & 0 \\ 0 & 0 & -b - \Lambda \end{vmatrix} = 0, \quad \text{or}$$

$$(b + \Lambda)[\Lambda^2 + (\sigma + 1)\Lambda + \sigma(1 - r)],$$

which has the eigenvalues:

$$\Lambda_1 = -b \quad \text{and}$$

$$\Lambda_{2,3} = \frac{1}{2} \left[-(\sigma + 1) \pm \sqrt{(\sigma - 1)^2 + 4r\sigma} \right].$$

These eigenvalues are real and negative for $0 < r < 1$ and the fixed point O_1 is stable. It is an hyperbolic attractor point of the system, and all the phase orbits end up at this. But $r < 1$, for the laser equations, means $\lambda < 0$, and there is

no laser action in this case. For $r=1$, i.e. for $\lambda=0$, the eigenvalues are: $\Lambda_1 = -b < 0$, $\Lambda_2 = 0$ and $\Lambda_3 = -(\sigma+1) < 0$. The fixed point O_1 becomes marginal stable. For $r > 1$, i.e. for $\lambda > 0$, the eigenvalues are: $\Lambda_1 = -b < 0$, $\Lambda_2 > 0$, $\Lambda_3 < 0$, and O_1 becomes a saddle point. Now O_1 has both a two-dimensional stable manifold, associated with the negative eigenvalues, and a one-dimensional unstable manifold associated with the positive eigenvalue.

When $r = 1$, the system goes through a pitchfork bifurcation whereby O_1 loses its stability, while the two fixed points $O_{2,3}$ are created. Thus when $r > 1$, i.e. when $\lambda > 0$, we get another two equilibrium states of the system, $O_{2,3}$. Now we can take into consideration the laser action. The stability of these fixed points is analysed with the same method as for the first fixed point.

The corresponding characteristic equation is:

$$\Lambda^3 + (b + \sigma + 1)\Lambda^2 + b(\sigma + r)\Lambda + 2b\sigma(r - 1) = 0 \quad (11)$$

Because $2b\sigma(r - 1) > 0$, the three solutions of the equation (10) verify the relation:

$$\Lambda_1 \Lambda_2 \Lambda_3 = -2b\sigma(r - 1) < 0. \quad (12)$$

In this case we have the following possibilities:

- all the roots are real, but one is negative and the others have the same sign;
- one root is negative and the two others are complex conjugates.

The fact that one eigenvalue is always negative, it means that there will always be at least a one-dimensional stable manifolds of $O_{2,3}$. When the others two eigenvalues are real and negative or have negative real parts, in a particular domain of the parameter r , the fixed points $O_{2,3}$ are stable foci. The typical phase space orbits are attracted by these fixed points along the direction corresponding to the directions of the stable manifold. At the same time the orbits spiral inwards around the fixed points approximately in the plane spanned by the eigenvectors corresponding to the complex conjugated set of eigenvectors.

As r increases, there is a critical value of it, r_c , when the complex conjugate eigenvalues have only the imaginary part. This value of the parameter r can be calculated by the following considerations [12]. Let us reconstruct a third order equation with the solutions: Λ_1 and $\Lambda_{2,3} = \pm i\Omega$, i.e.:

$$\Lambda^3 - \Lambda_1 \Lambda^2 + \Omega^2 \Lambda - \Lambda_1 \Omega^2 = 0, \quad (13)$$

which we compare with the equation (10). We obtain:

$$\Lambda_1 = -(\sigma + b + 1) \text{ and } \Omega = \pm \sqrt{b(\sigma + r_c)} \quad (14)$$

Then from $\Lambda_1 \Omega^2 = 2b\sigma(r_c - 1)$ we get:

$$r_c = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \quad (15)$$

and $r_c > 0$ only when $\sigma > b + 1$.

At $r = r_c$ there is a Hopf bifurcation and a chaotic attractor appears. For $r \geq r_c$, the equilibrium states $O_{2,3}$ become unstable as the real parts of the complex conjugate eigenvalues pass from negative to positive and there is only a chaotic attractor.

It was pointed out by Kaplan and Yorke [13], in the case of Lorenz system, that there is a transient chaos, (not attracted) for a range of values of $r < r_c$, and this transient chaos is converted to a chaotic attractor by a crisis at $r = r_1 < r_c$. There is also another important event on the route to chaos – the occurrence of a homoclinic orbit. This orbit separates the domain between the stable behaviour of the system and the transient chaos. The homoclinic orbit starts out along the unstable manifold of the unstable fixed point O_1 , makes just one swing around one of the points $O_{2,3}$, depending on the direction taken on the unstable manifold, and returns to O_1 .

When $r > r_c$ the system space phase is dominated by chaotic orbits with sensitive dependence on initial conditions. Any orbit that comes near the now unstable two-dimensional manifold of $O_{2,3}$ will spiral outwards because of the positive real part of the eigenvalues of $O_{2,3}$. But the particular form of the nonlinearity of the system prevents the orbit from wandering off to infinity. The orbit, sooner or later, comes into vicinity of the stable manifold of the other fixed point, and thus gets attracted towards the other focus where it again spirals outwards a number of times before it is attracted to the first focus, and so on. In this way the chaotic attractor is born. This attractor is strange because its geometry is a fractal one.

The considerations above presented can be converted to the case of laser equations. Thus, as the pump power λ is increased, when $\lambda \geq \lambda_c$, the chaotic behaviour could be observed. The value of λ_c can be obtained from (13) and (7):

$$\lambda_c = \frac{k(k + \gamma_{\parallel} + 3\gamma_{\perp})}{\gamma_{\perp}(k - \gamma_{\parallel} - \gamma_{\perp})} - 1 = \frac{(k + \gamma_{\perp})(k + \gamma_{\parallel} + \gamma_{\perp})}{\gamma_{\perp}(k - \gamma_{\parallel} - \gamma_{\perp})} \quad (16)$$

with $k > \gamma_{\parallel} + \gamma_{\perp}$.

4. Numerical results

The route to chaos in LRH system was numerically investigated for different values of the control parameter r (the pump power λ), and for the same parameters $\sigma = 2$,

$(\gamma_{\perp} = \frac{1}{2}, k = 1)$ and $b = \frac{1}{4}, (\frac{\gamma_{\parallel}}{\gamma_{\perp}} = \frac{1}{4})$, which are

appropriate for NH_3 laser [4].

The calculations was performed using the fourth order Runge-Kutta method and the results are illustrates in the Figs. 1- 5. In these figures, for each value of the parameter r , a 3-dimensional representation of the phase portrait of the system is illustrated, and also the evolution in time of the variable x .

The behaviour of the system for $r < r_c = 14$ ($\lambda_c = 13$) is shown in Figs. 1(a), 1(b) and Figs. 2 (a), 2(b). The system reaches one of the equilibrium points $(-1, -1, 1)$ or $(1, 1,$

1), depending on the choice of the initial conditions in the corresponding basin of attraction.

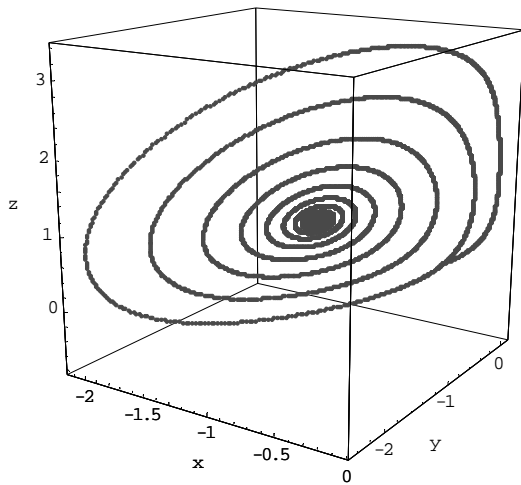


Fig. 1(a). The phase trajectory of the system, for $r = 5$ and the initial conditions $(-0.5, 0.1, 0.1)$, goes to the equilibrium point $(-1, -1, 1)$.

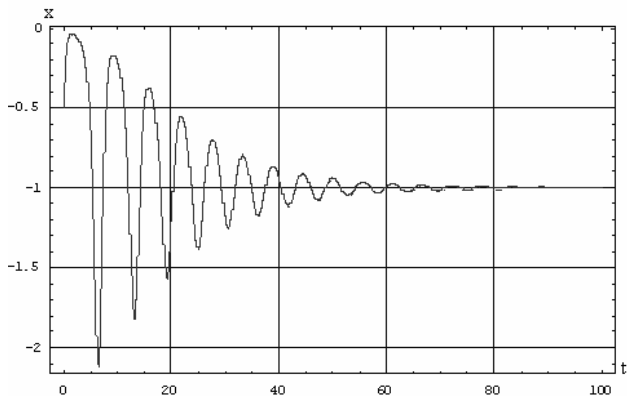


Fig. 1(b). The evolution in time of the variable x , for $r = 5$ and the same initial conditions as in Fig. 1(a).

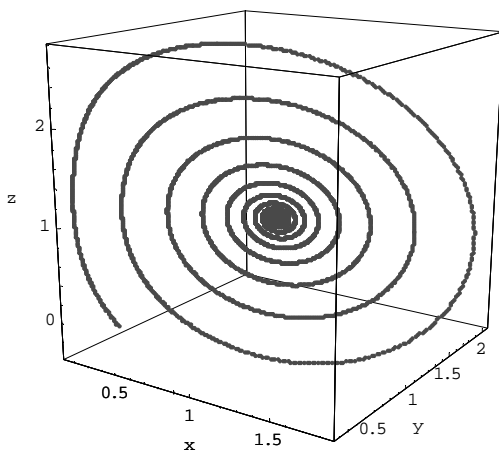


Fig. 2(a). The phase trajectory, for $r = 5$, but for the initial conditions $(0.5, 0.1, 0.1)$ reaches the other equilibrium point of the system, i.e. $(1, 1, 1)$.

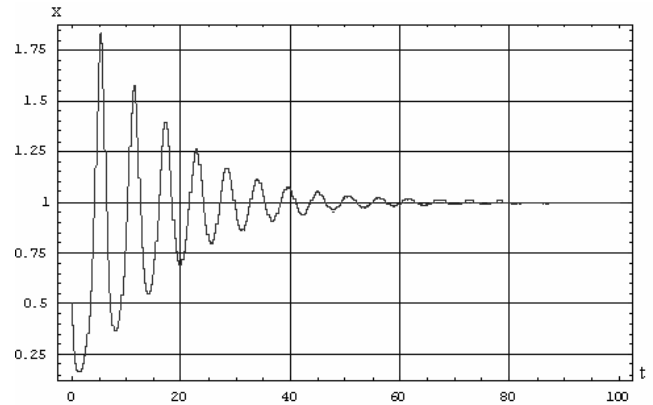


Fig. 2(b). The evolution in time of the variable x , for $r = 5$ and the same initial conditions as in Fig. 2(a).

For $r > r_c$ the behaviour of the system is a chaotic one. This is illustrated in the Figs. 3-5. In the Figs. 3(a) and 3(b) the value of the parameter r is 14.005, just above $r_c = 14$. In the Fig. 4 and Fig. 5 the value of the parameter r was considered much greater than r_c , i.e. 30 and 100, respectively. The chaotic behaviour of the system is obvious, and the 3-dimensional phase portraits are strange attractors.

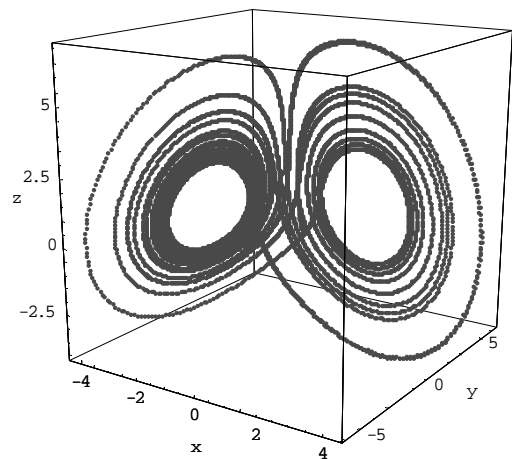


Fig. 3(a). A 3-dimensional phase portrait of the LRH system, for $r = 14.005$ and the initial conditions $(0.5, 0.1, 0.1)$.

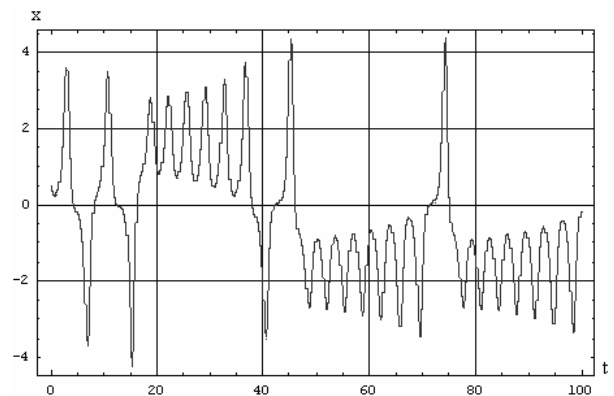


Fig. 3(b). The evolution in time of the variable x for $r = 14.005$ and the initial conditions $(0.5, 0.1, 0.1)$.

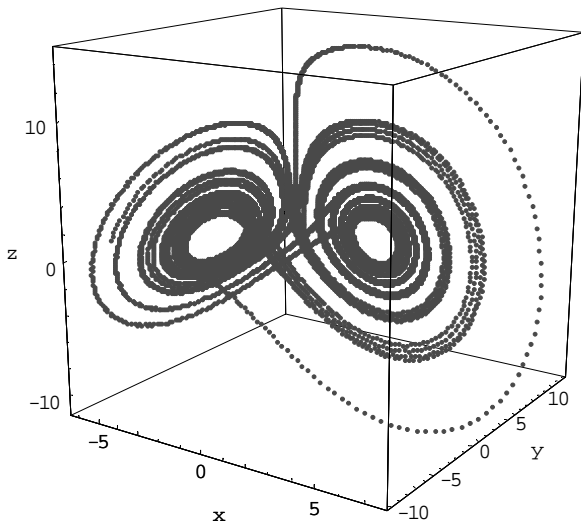


Fig. 4(a). A 3-dimensional phase portrait of the LRH system, for $r = 30$ and the initial conditions $(0.5, 0.1, 0.1)$.

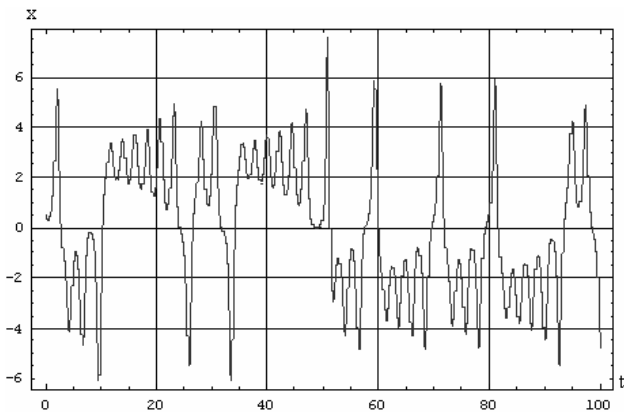


Fig. 4(b). The evolution in time of the variable x for $r = 30$ and the initial conditions $(0.5, 0.1, 0.1)$.

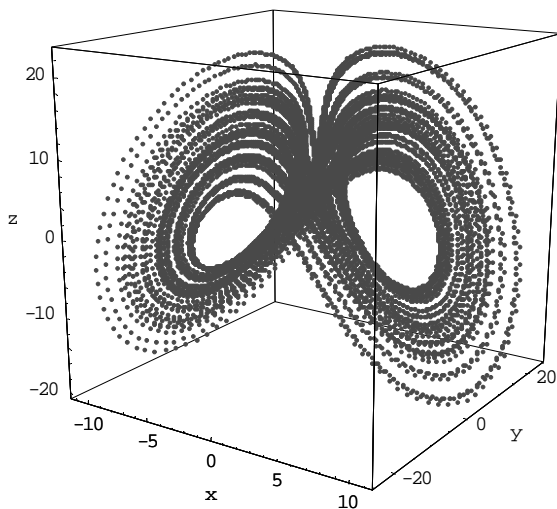


Fig. 5(a). A 3-dimensional phase portrait of the LRH system, for $r = 100$ and the initial conditions $(0.5, 0.1, 0.1)$.

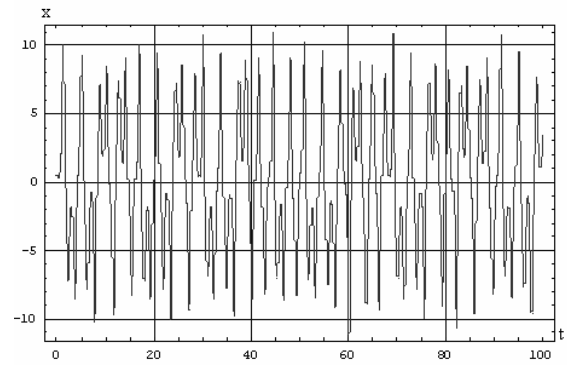


Fig. 5(b). The evolution in time of the variable x for $r = 100$ and the same initial conditions as in the Fig. 5(a).

5. Conclusions

In this work an analogy between Lorenz-Robbins system for a convective fluid flow and Haken equations for a single mode laser is presented. The system called Lorenz-Robbins-Haken (LRH) is both analytically and numerically investigated. As the control parameter, which is proportional with the pump power, is varied, the chaotic behaviour of the system is observed. This behaviour corresponds to the experimental observations [14-16].

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