

A system theoretic approach to array processing

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In this paper, utilizing system theoretic concepts a sound, rigorous theory of array processing is established which leads to several new results. MUSIC, MIN-NORM, ESPRIT, and PISARENKO used for both spatial and temporal spectral decomposition of signals are well known techniques in array processing. In this work, a general approach generalizing them is presented. A theory for multipath case is provided for analysis and design of array structures without the assumption of linearity and equal spaceness which estimates the temporal frequency and the directions for coherent sources. Our approach is also developed to null signals in certain directions with certain frequencies, such as for multipath cancellation.

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1. Introduction

Sensor array processing focuses on data collected at the sensors to carry out a given estimation task. In this paper, this array processing is developed via a system approach. Consider the situation where an array of sensors receive signals from multiple emitters. For many cases of interest (e.g., Radar, Sonar, Seismic Waves) the array output can be modeled as a State-Space model. The following equation can be written to represent a general case of the array system as the following [1 – 9].

$$\begin{aligned} x_i(t+1) &= F_i x_i(t), \quad i = 1, \dots, n \\ y_i(t) &= H_i x_i(t) + v_i(t) \\ x(t_0) &= x_0 \in C^n; \quad t = t_0, t_0 + 1 \end{aligned} \quad (1)$$

where continuous time index as t can be used as well as discrete case [27]. $x_i(t)$ is the state vector corresponding to the i -th sensor. F_i is the transition matrix for the i -th sensor, respectively. $y_i(t)$ is the measurement vector received by the i -th sensor. H_i is the measurement matrix corresponding to the i -th sensor. $v_i(t)$ is system noise associated with the targets seen by the i -th sensor, assumed to be normally distributed with the zero mean, and mutually uncorrelated. $x(t_0) = x_0$ is determined as exponentials of phases of incoming signals at t_0 . For example, if the emitted signals are sinusoidal, and the array is a linear equispaced one, then the components of (1) will be taken as,

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad F = \text{diag}(e^{j\omega_1 \Delta t}) = \begin{bmatrix} e^{j\omega_1 \Delta t} & 0 & \dots & 0 \\ 0 & e^{j\omega_2 \Delta t} & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & e^{j\omega_n \Delta t} \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{bmatrix} \\ y(t) &= \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \omega_1 & \omega_2 & \dots & \omega_n \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^n & \omega_2^n & \dots & \omega_n^n \end{bmatrix}, \quad v(t) = \begin{bmatrix} v_1^{p-1}(t) \\ v_2^{p-1}(t) \\ \vdots \\ v_p^{p-1}(t) \end{bmatrix} \end{aligned} \quad (2)$$

where p is the number of the elements in the sensor array and n is the number of targets (emitters). This model can describe the time of evolution of a signal through an array (Δt = time sampling interval), as well as the spatial evolution of delay along certain directions in a multi-dimensional array at a certain time, or a combination of both. How this formulation can be obtained for multidimensional arrays will be explained in Section 3. The parameters used in (2) are as

$$\begin{aligned} \omega_i &= e^{j\omega_i \zeta_i} \\ \omega_i &= i - \text{th emitter's frequency (the frequency of the signal incoming from } i - \text{th target)} \\ \zeta_i &= \text{delay due to angular location of the } i - \text{th emitter, in one direction} \\ x_{0i} &= \text{a complex number representing the range and amplitude of the sinusoidal signal from the } i - \text{th emitter} \end{aligned} \quad (3)$$

In the proposed method, the problem is to find ω_i 's and ζ_i 's or $\Delta \zeta$ as well.

MUSIC, MIN-NORM, ESPRIT, and PISARENKO are well known of the techniques used for both spatial and temporal spectral decomposition of signals in array processing [10 – 26]. For a standard exposition of the state-space and realization theory utilized here, is presented in [3]. Most literature approaches the problem of direction-frequency finding by treating each data point without much regard to its intrinsic evolution structure, as

a sample from a stochastic process. Data points are primarily viewed as tools for spatial or temporal averaging for estimation of certain covariances (e.g., [6 – 9, 13 – 26] and their references).

Although a state-space formulation of sinusoids evolving in time is considered in [1] initially, by immediately passing to covariances, the intrinsic structure is again lost substantially, working effectively with a state-space model of the covariances rather than the target itself. A quite different approach is taken where the state-space model of the signal itself is considered throughout, and the averaging operations are taken into consideration as forming reachability and observability grammians [3] directly, mainly as a substitute for covariances.

Our results here provide a qualitative and quantitative theory for explaining the effects of the structure of a multidimensional array in regards to modifying the ranks of certain matrices utilized in direction-frequency estimation. This is important for many significant sensor problems which arise in Radar, Sonar, Seismic Arrays, such as multipath and other situations where the same frequency sinusoids may arrive from different unknown directions. The key concepts in this theory are observability and controllability (reachability) [27]. Section 3 establishes this general theory for multidimensional arrays. As a consequence, in Section 4, we establish (easily) new results on direction cancellation (such as for multipath) via a spatial filter, and in Section 5 we establish a general techniques which unifies and extends techniques such as MUSIC and ESPRIT (e.g., [6 – 9, 13 – 26]).

Our approach also enables us to utilize a new type of correlation (explained in Section 2) which is dual to the commonly used one, and which proves complementary and quite useful as explained in Section 3. Thus, utilizing the system theoretic concepts a sound, rigorous theory of array processing is established which leads to several new results.

2. A new use of output correlations for frequency-direction estimation

In this paper we also use a new type (in order) correlation than the usual type used in MUSIC-PISARENKO techniques. First, we briefly explain this, as it is used in the paper for direction finding. For simplicity of exposition, our presentation will be in time versus one spatial variable though it will be replaced by another spatial variable along an array. Define:

$$Y_{t_0,m} := [y(t_0), y(t_0 + 1), \dots, y(t_0 + m)] \quad (4)$$

$$V_{t_0,m} := [v(t_0), v(t_0 + 1), \dots, v(t_0 + m)] \quad (5)$$

Then for some integer $\sigma \geq 0$, consider the correlation,

$$Y_{t_0,m}^* Y_{t_0+\sigma+m,m} = \begin{bmatrix} x_0^* \\ x_0^* F^* \\ \vdots \\ x_0^* F^{*m} \end{bmatrix} H^* H F^{\sigma+m} [x_0, \dots, F^m x_0] + \begin{bmatrix} x_0^* \\ x_0^* F^* \\ \vdots \\ x_0^* F^{*m} \end{bmatrix} H^* V_{t_0+\sigma+m,m} + V_{t_0,m}^* H F^{\sigma+m} [x_0, \dots, F^m x_0] + V_{t_0,m}^* V_{t_0+\sigma+m,m} \quad (6)$$

The following assumptions are true in almost all cases of arrays of sensors.

A1: There exist an integer $\sigma \geq 1$ such that $E\{v^*(t)v(t+\zeta)\} = 0$ for all ζ and $|\zeta| \geq \sigma$

A2: $E\{v(t)\} = 0$

A3: The matrix F is nonsingular

A4: $p \geq m$, and the columns of H are linearly independent over C

Under these assumptions, we obtain

$$K_{t_0,\sigma,m} := E\{Y_{t_0,m}^* Y_{t_0+\sigma+m,m}\} = \begin{bmatrix} x_0^* \\ x_0^* F^* \\ \vdots \\ x_0^* F^{*m} \end{bmatrix} H^* H F^{\sigma+m} [x_0, F x_0, \dots, F^m x_0] \quad (7)$$

By the assumptions A3 and A4, $K_{t_0,\sigma,m} = \ker[x_0, F x_0, \dots, F^m x_0]$ (where, for matrix A , $\ker A = \text{kernel of } A = \text{null space of } A$). Note that $\ker K_{t_0,\sigma,m}$ is an $(m+1) \times (m+1)$ matrix. Now it is easy to infer:

Theorem 2.1 The smallest value of m for which $\text{rank } \ker K_{t_0,\sigma,m} \leq m+1$ is the reachability index, n_0 of (F, x_0) .

In such a case $\text{rank } \ker K_{t_0,\sigma,m} = m = n_0$. If we choose the unique numbers $a_0, a_1, \dots, a_{n_0-1} \in C$ such that

$$k_{n_0} = \sum_{j=0}^{n_0-1} a_j k_j \quad (8)$$

$$F^{n_0} x_0 = \sum_{j=1}^{n_0-1} a_j F^j x_0 \quad (9)$$

the polynomial

$$a(z) = a_0 + a_1 z + \dots + a_{n_0-1} z^{n_0-1} + z^{n_0} \quad (10)$$

is the characteristic polynomial of the matrix F_1 where (F_1, x_{01}) is the reachable part of (F, x_0) . The roots of $a(z)$ are $e^{j\omega_i \Delta t}$'s. For time-averages used in place of expected values (with a purely deterministic approach), we consider

$$\sum_{t=t_0}^{t_0+M} Y_{t,m}^* Y_{t_0+\sigma+m,m} \quad (11)$$

Then, the assumptions A1–A3 translate into the sum of the last three terms over $[t_0, t_0 + M]$ being negligibly small. Then we work with

$$K_{t_0, \sigma, m} := \Omega K \Omega^* \quad (12)$$

where

$$\Omega = \begin{bmatrix} x_0^* \\ x_0^* F^* \\ \vdots \\ x_0^* F^{*m} \end{bmatrix} \quad (13)$$

$$K = \sum_{k=0}^M F^{*k} H^* H F^{k+\sigma+m} \quad (14)$$

the same way as we did with $K_{t_0, \sigma, m}$. Note that K is the product of the observability grammian of (F, H) with $F^{\sigma+m}$. Thus, even if all ω_i 's may be the same, distinct ω_i 's can still be determined.

Remark 2.1

Note that the usual type of correlations utilized are of the type

$$E\{YY^*\} \quad (15)$$

$$E\{Y^*Y\} \quad (16)$$

which involves the observability grammian. With the unifying approach in [1] and here, these are both meaningful and dual, whereas (15) would not have a clear interpretation with the usual way or treating $y(t)$ simply as data (ignoring its internal structure). Thus, whatever applies in terms of the controllability grammians in Sections 3-5, using (15) type correlations also applies in

terms of the observability grammians using the new (16) type correlations.

3. Use of spatial structures in arrays for direction estimation

In some situations such as in multipath, the signals from several emitters can be correlated. For example, all the signals may have the same frequency, but the angles of arrival may be different. In such cases the reachability grammian used in techniques which are generalizations of PISARENKO's technique, such as those in [27] and its references, will have a lower rank than number of signals, and the directions of the signals cannot be determined (since, then, we will identify a signals model (F, H, x_0) whose dimension is less than the number of emitters). In such cases the spatial geometry of the array of sensors can be arranged to alleviate this problem. The following provides a qualitative and quantitative theory for this purpose.

Suppose that we have an array with some arbitrary geometry, and n emitters each from a different direction emit the same frequency sinusoid $e^{j\omega_k \Delta t}$, $k = 0, 1, 2, \dots$. The signal model for a single emitter will be

$$(x+1) = e^{j\omega \Delta t} x(t); x(0) = A_1 = \text{amplitude and the phase of the emitted signal}$$

$$y_1(t) = \begin{bmatrix} 1 \\ e^{-j\omega \zeta_1} \\ e^{-j\omega \zeta_2} \\ \vdots \\ e^{-j\omega \zeta_{p-1}} \end{bmatrix} x(t) + v(t); t = 0, 1, 2, \dots$$

noise

$$(17)$$

where the delays ζ_i 's are determined by the geometry of the array and the angle of the emitter relative to the array. Suppose there is another emitter emitting $e^{-j\omega_k \Delta t}$ from another direction. The total received signal is

$$y_1(t) = \begin{bmatrix} 1 & 1 \\ e^{-j\omega \zeta_1} & e^{-j\omega \gamma_1} \\ \vdots & \vdots \\ e^{-j\omega \zeta_{p-1}} & e^{-j\omega \gamma_{p-1}} \end{bmatrix} x(t) + v(t) \quad (18)$$

$$x(t+1) = \begin{bmatrix} e^{-j\omega \Delta t} & 0 \\ 0 & e^{-j\omega \Delta t} \end{bmatrix} x(t); x(0) = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

amplitude and phase at $t=0$

γ_i 's will be different from ζ_i 's as the angles are different. Thus, the signal model (F, H, x_0) is observable. However, it is not reachable. The reachable and observable part of the signal model has dimension one. Thus, no technique can identify the true (F, H, x_0) from $Y_1(t)$ via covariance techniques where the signal covariance is

(essentially the controllability grammian (see [3]). We can only identify the one-dimensional reachable and observable part of this signal.

Next suppose that we use another arbitrary array with the same number of elements and we pair the elements of the two arrays in some (arbitrarily chosen) manner, only subject to the constraint that each element of the second array receives a signal (from any fixed direction) with the same amount of delay, ϕ , relative to one (different) element of the first array. ϕ will depend on the geometry of the array as well as the direction of the emitter. (An example of such an array is a planar rectangular array). thus, the output of the second array will be

$$y_2(t) = H_1 F_1 x(t) \quad (19)$$

$$F_1 = \begin{bmatrix} e^{-j\omega\phi} & 0 \\ 0 & e^{-j\omega\phi_2} \end{bmatrix} \quad (20)$$

where ϕ_i is the delay associated with each emitter. Then we can write

$$Y(t) = [y_1(t) \quad y_2(t)] = H_1 F' [x_0 \quad Fx_0] + [v_1(t) \quad v_2(t)] \quad (21)$$

where $v_1(t), v_2(t)$ are noise processes associated with each array. Thus, for any $t \geq 0$, we have

$$E\{Y(t)Y^*(t)\} = H_1 F' [x_0 \quad Fx_0] \begin{bmatrix} x_0^* \\ x_0^* F_1^* \end{bmatrix} F^* + \sigma.I \quad (22)$$

where

$$\sigma.I = E \left\{ \begin{bmatrix} v_1(t) & : & v_2(t) \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \right\} \quad (23)$$

where we assumed that $v_1(t)$ and $v_2(t)$ are spatially white processes (i.e., the measurement noise process for each subarray is uncorrelated ;

If, in general, there are n emitters with frequencies $\omega_1, \dots, \omega_n$ (some subsets of the ω 's can be identical) , and if we use n arrays each containing p sensors with outputs y_1, \dots, y_n , arranged such that any two succeeding arrays are placed identically with respect to relative delays, and as described above for two arrays (such as a rectangular array equally spaced in one direction), then

$$Y(t) = [y_1(t) \quad y_2(t)] = HF' [x_0, F_1 x_0, \dots, F_1^{n-1} x_0] + [v_1(t), \dots, v_n(t)] \quad (24)$$

where

$$H = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{-j\omega_1 \zeta_1^1} & e^{-j\omega_2 \zeta_1^2} & \dots & e^{-j\omega_n \zeta_1^n} \\ e^{-j\omega_1 \zeta_2^1} & e^{-j\omega_2 \zeta_2^2} & \dots & e^{-j\omega_n \zeta_2^n} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-j\omega_1 \zeta_{p-1}^1} & e^{-j\omega_2 \zeta_{p-1}^2} & \dots & e^{-j\omega_n \zeta_{p-1}^n} \end{bmatrix} \quad (25)$$

$$F_1 = \begin{bmatrix} e^{j\omega_1 \phi} & 0 & \dots & 0 \\ 0 & e^{j\omega_2 \phi_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{j\omega_n \phi_n} \end{bmatrix} = \text{diag}\{e^{j\omega_i \phi_i}\}_{i=1}^n \quad (26)$$

$$x_0 = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}; A_i = \text{amplitude and the phase of the signal from the } i\text{-th emitter} \quad (27)$$

$$F = \begin{bmatrix} e^{j\omega_1 \Delta t} & 0 & \dots & 0 \\ 0 & e^{j\omega_2 \Delta t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{j\omega_n \Delta t} \end{bmatrix} = \text{diag}\{e^{j\omega_i \Delta t}\} \quad (28)$$

More generally, for $t \geq 0, k \geq 0$, we can consider both temporal and spatial time shift situations simultaneously by considering

$$Y_{t,k} = \begin{bmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+k) \end{bmatrix} \quad (29)$$

$$Y_{t,k} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^k \end{bmatrix} F' [x_0, F_1 x_0, \dots, F_1^{n-1} x_0] + \begin{bmatrix} v_1(t) & v_2(t) & \dots & v_n(t) \\ v_1(t+1) & v_2(t+1) & \dots & v_n(t+1) \\ \vdots & \vdots & \vdots & \vdots \\ v_1(t+k) & v_2(t+k) & \dots & v_n(t+k) \end{bmatrix} \quad (30)$$

Then,

$$E\{Y_{t,k} Y_{t+\sigma, \ell}^*\} \quad (31)$$

for any suitable integers t, k, σ, ℓ , so that the noise covariance term in (31) is either diagonal or zero; to obtain expressions of the type,

$$\sum_{k,l,\sigma,t} = E\{Y_{t,k} Y_{t+\sigma,l}^*\} = \underbrace{\begin{bmatrix} H \\ HF \\ \vdots \\ HF^t \\ HF^k \end{bmatrix}}_{H_i} F^t \underbrace{[x_0, F_1 x_0, \dots, F_1^{n-1} x_0]}_{\Omega_1} \Omega_1^* F^{t+\sigma} H_i^* + Q_1 + \begin{bmatrix} v_1(t) & v_2(t) & \dots & v_n(t) \\ v_1(t+1) & v_2(t+1) & \dots & v_n(t+1) \\ \vdots & \vdots & \vdots & \vdots \\ v_1(t+k) & v_2(t+k) & \dots & v_n(t+k) \end{bmatrix} \quad (32)$$

where Q_1 represents the noise covariance term. If σ is chosen to be the finite decorrelation time for the noise processes, then $Q_1 = 0$. If $\sigma = 0, k = \ell$ and, noise is spatially white, and temporally white Q_1 will be of the form μI , for some scalar $\mu \geq 0$. We can also form covariances of the form

$$\{Y_{t,k} Y_{t+\sigma,k}^*\} = \Omega_1^* F^{*t} H_k^* H_k F^{t+\sigma} \Omega_1 + Q_2 \quad (33)$$

where Q_2 is zero of $\sigma = \text{decorrelation interval for the noise process}$.

Under suitable whiteness assumptions of $v_i(t)$, it will be of the type $\rho I (\rho \geq 0 \text{ scalar for } \sigma = 0)$. Note that from (32) and (33) one can obtain ω_i 's and the two sets of direction cosines separately. For direction finding with ω_i 's known, if there are enough many sensors in the array direction forming H so that the columns of H are linearly independent, then it is enough to use (32) and (33) with $k = 0$.

The important point is that via the use of more arrays, now we have ensured that (F_1, x_0) is reachable, even though (F, x_0) may not be which is the condition for the covariances to have the necessary rank for frequency or direction determination (e.g., in [27] using a dimensional array).

By using H_k for suitable k instead of only H , we can apply these techniques to the cases where the columns of H are not necessarily independent (say there are less sensors, ρ , in each array, than the number of emitters, n).

Still the columns of H_k can be linearly independent while those of H are not (for determining ω_i 's as in [27]).

Remark 3.1

Similarly, one can consider

$$Y_{t,\ell} = \begin{bmatrix} y_1(t) & y_1(t+1) & \dots & y_1(t+\ell) \\ y_2(t) & y_2(t+1) & \dots & y_2(t+\ell) \\ \vdots & \vdots & \vdots & \vdots \\ y_n(t) & y_n(t+1) & \dots & y_n(t+\ell) \end{bmatrix} \quad (34)$$

and take the cross covariances as above. Now the expressions obtained will be similar to (32) and (32) with the only difference being that F and F_1 will be exchanged in the resulting expressions. Still the same techniques can be used to find F, F_1, x_0 , and H . Finally, note that our remarks in Section 5 also apply in this situation.

Remark 3.2

As explained in [27], the results developed here for an array of sensors can also be applied to any structure or medium for *i)* determining the dominant modes of vibrations due a disturbance, and *ii)* for locating the direction from which the disturbance propagates along the structure.

Remark 3.3

From the results developed in this paper so far, it is clear at this point how to arrange data points into matrices in certain ways and cross-correlate them (to cancel the terms due to noise) to obtain expressions of the type

$$\underbrace{\begin{bmatrix} H \\ HF \\ \vdots \\ HF^k \end{bmatrix}}_{L_1} F^t [x_0, F_1 x_0, \dots, F_1^m x_0] \cdot L_1^* \quad (35)$$

or

$$\underbrace{\begin{bmatrix} H \\ HF_1 \\ HF_1^2 \\ \vdots \\ HF_1^k \end{bmatrix}}_{L_2} F^t [x_0, F_1 x_0, \dots, F_1^m x_0] \cdot L_2^* \quad (36)$$

or

$$\underbrace{\begin{bmatrix} H \\ HF \\ \vdots \\ HF_k \end{bmatrix}}_{L_3} F^t F^t [G_0^m, F_1 G_0^m, \dots, F_1^\ell G_0^m] \cdot L_3^* \quad (37)$$

(where $G_0^m = [x_0, F_1 x_0, \dots, F_1^m x_0]$), or

$$\underbrace{\begin{bmatrix} H_0 \\ H_0 F \\ \vdots \\ H_0 F^k \end{bmatrix}}_{L_4} F^t [G_0^m, F_1 G_0^m, \dots, F^\ell G_0^m] L_4^* \quad (38)$$

(where $H_0 = \begin{bmatrix} H \\ HF_1 \\ \vdots \\ HF_1^s \end{bmatrix}$), and possibly some other

expressions (for example, some or all of the F 's and F_1 's can appear interchanged). t, k, m, ℓ , and s denote some nonnegative integers. In (35)-(38) for a given $L_i (1 \leq i \leq 4)$, L_i denotes a same type of matrix with possibly different values for the integers occurring in L_i , depending on the situation (while checking linear dependencies on the rows or columns of the correlations).

Now one can use the same type of eigenvalue-eigenvector techniques or the ones in [1] and its references to obtain frequencies (via eigenvalues of F) and/or direction of arrivals (via eigenvalues of F_1 and H).

Remark 3.4

By correlating the L 's in a different order one can also obtain expressions of the type

$$L_1^* L_1 L_2^* L_2, \dots, L_4^* L_4 \quad (39)$$

where, now the conjugate transposes of the reachability type matrices replace the observability type matrices occurring in (35)-(38). Note that this type of correlation has not been used in the literature before, although, it is seen to be quite useful from the theory developed here.

These forms give rise to several possible techniques of direction of arrival estimation. Some examples are:

i) Form $L_1^* L_1$ to obtain, say,

$$L_1^* L_1 = \underbrace{\begin{bmatrix} x_0^* \\ x_0^* F_1^* \\ \vdots \\ x_0^* F_1^{*m} \end{bmatrix}}_{G_0^*} F^{*t_1} \underbrace{[H^*, F^* H^*, \dots, F^{*k} H^*]}_{H_k^*} H_k F^{t_2} G_0 \quad (40)$$

Thus, by finding the linear dependencies among the rows or columns of $L_1^* L_1$, we can obtain F_1 (which usually describes one of the direction cosines). Then, forming

$$HF^{t_1} G_0 G_0^* F^{*t_2} H^* \quad (41)$$

and finding the linear dependencies, we can determine the remaining direction cosine.

ii) To obtain both angles simultaneously, consider $L_2^* L_2$, and then, apply the technique of finding the linear dependencies among the rows or columns, as described in (35)-(38) and the references of these papers.

iii) Using $L_1^* L_1$ and $L_1 L_1^*$, one can obtain the frequencies (i.e., F) as well as angles of arrivals (from H and F_1).

Clearly, many variations of these techniques are possible. Note that with these techniques x_0 can also be found (which gives the amplitude and the phase of the signal from each direction).

Remark 3.5

We should note that in many real situations, there will be a continuum of angles of arrivals. (say, clutter signals for Radars, or Multipath of Sonar Signals). Thus, no matter how many sensors are used to form how many arrays, usually the number of rows H will be always less than its columns. Also, any linear dependencies will be only approximate. thus, linear dependencies must be determined using approximate techniques developed for this purpose (e.g., (35)-(38)).

4. Nulling signals from certain directions

The signals from certain directions (say multipath) can be canceled as follows. Let's consider the general expression (38) with $\ell = r = t = 0$, to obtain

$$H_0 G_0^m = \underbrace{\begin{bmatrix} H \\ HF_1 \\ \vdots \\ HF_1^s \end{bmatrix}}_{L_4^0} [x_0, F_1 x_0, \dots, F_1^m x_0] \quad (42)$$

where L_4^0 is a proper possibly shifted (temporally in terms of sampling time, and/or spatially in terms of the sensors) version of L_4^0 . Let $Q(z)$ be defined as

$$Q(z) = \begin{bmatrix} q_1(z) \\ q_2(z) \\ \vdots \\ q_n(z) \end{bmatrix} \quad (43)$$

where $q_i(z)$ is the i -th row of $Q(z)$, a polynomial vector, given as

$$q_i(z) = \sum_{j=0}^{v_i-1} q_{ij} z^j \quad (44)$$

where $v_i - 1$ is the observability index, and q_{ij} describe linear dependencies among the rows of the observability matrix of (H_1, F_1) in lexicographic order, (e.g., [1] and its references). Then the rows of $q_i(z)$ can be put together to form matrices Q_i such that

$$[Q_0, Q_1, \dots, Q_{v-1}, -I] \cdot \begin{bmatrix} H_1 \\ H_1 F_1 \\ \vdots \\ H_1 F_1^v \end{bmatrix} = 0 \quad (45)$$

where $v = \max\{v_i\}$ is the maximum of these indices (called the observability index), or,

$$[Q_0 : 0, Q_1 : 0, \dots, Q_{v-1} : 0, -I : 0] \cdot \begin{bmatrix} H_1 \\ H_1 F_1 \\ \vdots \\ H_1 F_1^v \end{bmatrix} = 0 \quad (46)$$

Thus, if we weigh the sensor array outputs according to (46), the sum will be zero for the signals which determine H and F_1 via their frequencies and angles (both the elevation and the azimuth). From (46) and $H_2 = T_1 H_1$ one can characterize all such weightings to cancel reception of signals characterized by (H, F_1) .

Remark 4.1

Usually we may not wish to cancel all the incoming signals. Then, we can decompose (H, F_1) into two parts such as

$$H = [H_{11} \quad H_{12}] ; F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} \quad (47)$$

so that the signals we wish to cancel are characterized by (H_{11}, F_{11}) , then we have the form

$$\begin{bmatrix} H \\ HF_1 \\ \vdots \\ HF_1^k \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{11} F_{11} & x \\ H_{11} F_{11}^2 & x \\ \vdots & \vdots \\ H_{11} F_{11}^k & x \end{bmatrix} \quad (48)$$

where x 's denote some possibly nonzero matrices. Then we can apply the above technique to (H_{11}, F_{11}) to cancel the signals corresponding to the direction and modes of F_1 determining H_{11} and F_{11} only.

5. A generalization of ESPRIT

A comparison of the techniques for direction estimation given in this section (starting with remark 3.5) versus another well-known techniques of direction estimation, ESPRIT (see e.g., [8], and the references there) is as follows. Later, it will be seen that the techniques given here provide an extension of ESPRIT. For ESPRIT two subarrays are utilized to obtain (ignoring noise)

$$\begin{bmatrix} H \\ HF_1 \end{bmatrix} S(t) \quad (49)$$

where the columns of $S(t)$ are the noise-free signal samples. Then the generalized eigenvalue problem for

$$H(\lambda I - F_1)S(t) \quad (50)$$

is solved to determine λ_i 's which will be the eigenvalues of F_1 provided that *i*) the rows of $S(t)$ are linearly independent, *ii*) the columns of H are linearly independent ; $p \geq n$.

In our techniques, with the type of cross correlations done as described in this section (for noise considerations) we search for (approximate) linear dependencies (observability indices) among the rows of

$$\begin{bmatrix} H \\ HF_1 \\ \vdots \\ HF_1^k \end{bmatrix} \quad (51)$$

where T is a nonsingular matrix. Under assumptions *i*) and *ii*) above, all the observability indices of the pair (H, F_1)

will be equal to one. If we partition H (after, perhaps rearranging the rows of H if necessary)

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}; H_2 = T_1 H_1 \quad (52)$$

for some matrix T_1 we have

$$\overline{F} H_1 T - H_1 F_1 T = 0 \quad (53)$$

That is, F_1 describes the linear dependencies. If, as in ESPRIT, we also assume that F_1 is diagonal,

$$F_1 = \begin{bmatrix} \omega_1 & \cdot & \cdot & 0 \\ 0 & \omega_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \omega_n \end{bmatrix}; \omega_i = e^{-j\omega_i \gamma_i} \quad (54)$$

Then, clearly the columns of H_1 are the eigenvectors of F_1 , with the corresponding eigenvalues ω_i 's. Thus, in this case, our general techniques of direction estimation yield a technique that is an alternative to ESPRIT, when there is no noise.

However, it should be noted that since in many cases there will be infinitely many angles of arrivals, in actuality, even to an approximation, H will contain more columns than rows no matter how many sensors are used.

F_1 will also contain infinitely many eigenvalues (that is, if we assume a discretization of the continuous angle range from which signals are received).

Thus, the approximate linear dependencies of the rows of $H F_1 T$ to those of $H T$ may not be very satisfactory. As we increase k and consider the approximate linear dependencies of the rows of

$$H_k T = \begin{bmatrix} H \\ H F_1 \\ \vdots \\ H F_1^k \end{bmatrix} \quad (55)$$

the rows of $H F_1 T$ will be more linearly dependent (approximate dependency), because then we will be assuming F_1 to contain more angular direction, and thus ignoring a lesser number of spatial modes. Thus, to obtain more satisfactory direction estimated, one must use a larger number of $(k+1)$ subarrays rather than constraining to two subarrays. Thus, then one can use the general techniques developed in this section to achieve

direction estimation to a greater degree of accuracy. This provides a generalization of ESPRIT.

Remark 5.1

If F_1 is diagonal, it is clear that the i -th column of $\omega_i H$ and $H F_1$ will be the same. Thus, then

$$\omega_i H - H F_1 \quad (56)$$

will have an identically zero column. Thus, this can be used as a technique to null the signals from an angle specified by one of ω_i 's direction. That is

$$[\omega_i I \quad -I] \begin{bmatrix} H \\ H F_1 \end{bmatrix} = 0 \quad (57)$$

More generally, if

$$p(z) = (z - \omega_1) \cdots (z - \omega_r) = p_0 + p_1 z + \cdots + p_r z^r; r \leq k \quad (58)$$

then

$$[p_0 I, p_1 I, \dots, p_r I] \begin{bmatrix} H \\ H F_1 \\ \vdots \\ H F_1^k \end{bmatrix} = H_p(F_1) \quad (59)$$

will have its $i_1 - th, i_2 - th, \dots, i_r - th$, columns to be zero. Thus, p_0, \dots, p_r are proper weightings to null the signals from r directions (specified by ω 's) using $r+1$ subarrays, regardless of H (the other angle). To achieve nulling from point directions however, H must also be taken into account.

Remark 5.2

An equally important use of an array with n or more subarrays is the following. For all the Spectral Decomposition, Direction Estimation techniques to work, the signal covariance matrix has to be assumed nonsingular. This is known not to be the case when emitter signals (such as the case of multipath) are correlated. Via our treatment of the signal as a deterministic totally unknown quantity and via the use of n subsensors, we are able to exploit the concept of reachability, being able to replace the signals covariance matrix via the reachability grammian of the pair (F, x_0) and/or the reachability grammian of the pair (F_1, x_0) . This is used to ensure the nonsingularity of the reachability grammian even if all the signals are of the same type (completely correlated), say,

$e^{j\omega_k \Delta t}$, with the same frequency. This also enables us, if necessary, to completely separate the estimation of F_1 and of H (two different direction cosines).

6. Conclusion

In this paper, a system theoretic approach is developed which extends techniques such as MUSIC, MIN-NORM, ESPRIT, and PISARENKO for both spatial and temporal spectral decomposition of signals. It is shown that use of system theoretic concepts leads to a sound, rigorous theory of array processing, which yields several new techniques.

Future work will concentrate on the use of the extrapolation of this theoretical study in order to obtain improvements in the implementation of the developed approaches.

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